# Time evolution of subsystems

## Initial system-environment correlations

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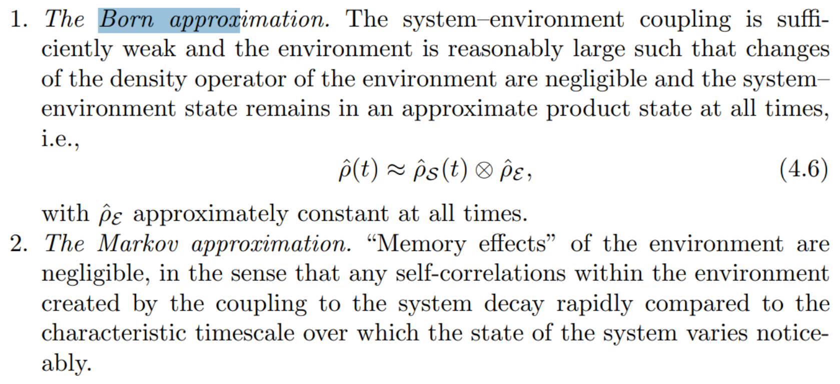
# Introduction

# Born and Markov approximations



1. *The Born approximation.* The system–environment coupling is sufficiently weak and the environment is reasonably large such that changes of the density operator of the environment are negligible and the system–environment state remains in an approximate product state at all times, i.e.,

$$\hat{\rho}(t) \approx \hat{\rho}_{\mathcal{S}}(t) \otimes \hat{\rho}_{\mathcal{E}}, \tag{4.6}$$

   with $\hat{\rho}_{\mathcal{E}}$ approximately constant at all times.
2. *The Markov approximation.* "Memory effects" of the environment are negligible, in the sense that any self-correlations within the environment created by the coupling to the system decay rapidly compared to the characteristic timescale over which the state of the system varies noticeably.

# Critique of the textbook presentation

- These are just words: *sufficiently week*, *reasonably large*. Where are the small or large parameters?
- If there is interaction, there is entanglement. How could the environment's state remain constant?
- *Approximate product state*, where is the small parameter describing it?
- What shoud I do if the memory effects are not-so-negligible?

**What is the next order in all these approximations?**

# P, TP, CP, . . .

What we have:

$$\rho_s(t) = \mathrm{Tr}_\epsilon \left( U_{t,0} \rho_{s\epsilon}(0) U_{t,0}^+ \right)$$

What we would like to have:

$$\rho_s(t) = V(t)\rho_s(0)$$

We would like to have a $V(t)$ such that it works for any $\rho_s$.

Positive: maps positive operators to positive operators.

Trace preserving: $\mathrm{Tr}\rho_s(0) = 1 \to \mathrm{Tr}\rho_s(t) = 1$.

Hermiticity preserving: $\rho_s(0) = \rho_s(0)^+ \to \rho_s(t) = \rho_s(t)^+$.

n-positivity: $V$ is n-positive if $V \otimes I_n$ is positive.

Positivity $\equiv$ 1-positivity

$N_s = \mathrm{dim} H_s$

$N_s$-positivity $\equiv$ complete positivity

# . . . divisibility

$V(t,s) = V(t)V^{-1}(s)$
If $V^{-1}(s)$ exists for all $s > 0$: $V$ is divisible.
$V(t,0) = V(t)$
$V(t,0) = V(t,s)V(s,0)$
P-divisible: $V(t,s)$ is positive.
CP-divisible: $V(t,s)$ is completely positive.

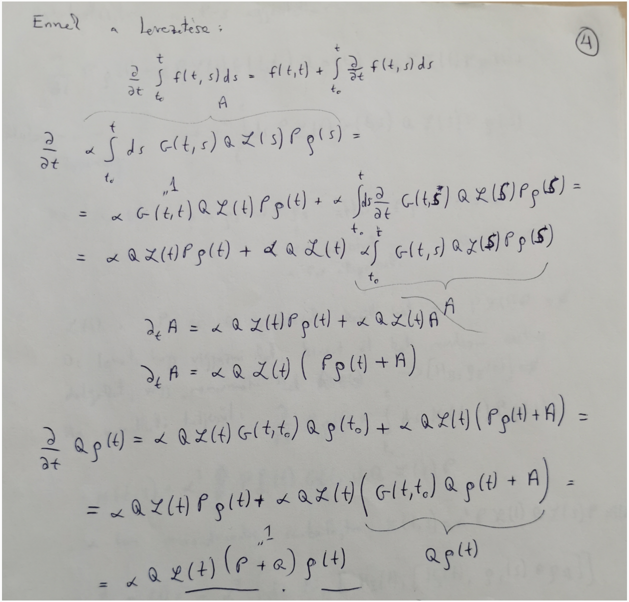
Positivity is a physical requirement.
Complete positivity is not really.

Ideology: if the time evolution operator is CP, we will get positive density matrices even if our system is entangled with an ancilla. So we should describe time evolution as a CP map.

Counterargument: it seems that sometimes the time evolution is non-CP. . .

# Nakajima-Zwanzig

# Details of the calculation [1]



Ennek a Levezetése: ④

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,s)\,ds = f(t,t) + \int_{t_0}^{t}\frac{\partial}{\partial t} f(t,s)\,ds$$

$$\frac{\partial}{\partial t}\overbrace{\alpha\int_{t_0}^{t} ds\; G(t,s)\,Q\,\mathcal{L}(s)\,P\rho(s)}^{A} =$$

$$= \alpha\,\overset{„1}{G(t,t)}\,Q\,\mathcal{L}(t)\,P\rho(t) + \alpha\int_{t_0}^{t} ds\,\frac{\partial}{\partial t}\,G(t,s)\,Q\,\mathcal{L}(s)\,P\rho(s) =$$

$$= \alpha\,Q\,\mathcal{L}(t)\,P\rho(t) + \alpha\,Q\,\mathcal{L}(t)\,\underbrace{\alpha\int_{t_0}^{t} G(t,s)\,Q\,\mathcal{L}(s)\,P\rho(s)}_{A}$$

$$\partial_t A = \alpha\,Q\,\mathcal{L}(t)\,P\rho(t) + \alpha\,Q\,\mathcal{L}(t)\,A$$

$$\partial_t A = \alpha\,Q\,\mathcal{L}(t)\left(P\rho(t) + A\right)$$

$$\frac{\partial}{\partial t}Q\rho(t) = \alpha\,Q\,\mathcal{L}(t)\,G(t,t_0)\,Q\rho(t_0) + \alpha\,Q\,\mathcal{L}(t)\left(P\rho(t) + A\right) =$$

$$= \alpha\,Q\,\mathcal{L}(t)\,P\rho(t) + \alpha\,Q\,\mathcal{L}(t)\underbrace{\left(G(t,t_0)\,Q\rho(t) + A\right)}_{Q\rho(t)} =$$

$$= \alpha\,Q\,\mathcal{L}\underline{(t)\,(P}\overset{„1}{+}Q)\,\rho(t)\;.\;\underline{\phantom{\rho(t)}}$$

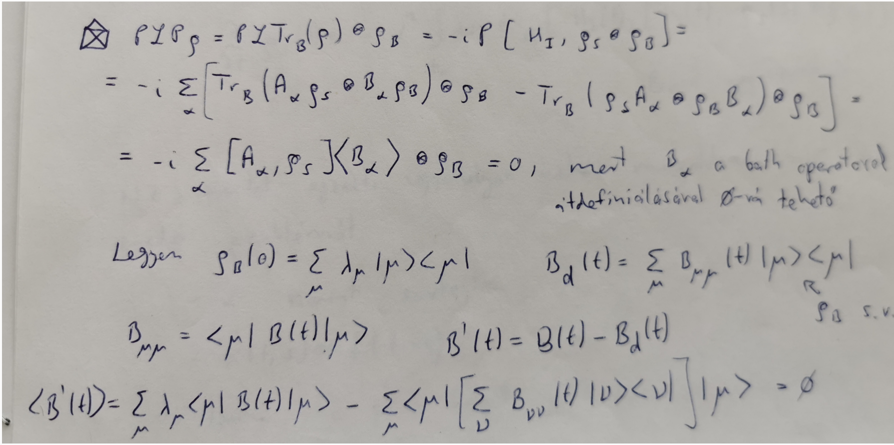
# Results

After solving the $Q$ part and putting it back to the important $P$ equation:

$$\begin{aligned}\frac{\partial}{\partial t}P\rho(t) &= \alpha P\mathcal{L}(t)G(t,t_0)Q\rho(t_0)+\alpha P\mathcal{L}(t)P\rho(t)+\\ &+ \alpha^2\int_{t_0}^{t}dsP\mathcal{L}(t)G(t,s)Q\mathcal{L}(s)P\rho(s)\end{aligned}$$

- There is nothing second order in the equation
- If there are no initial correlations, $\rho(t_0)=\rho_s(t_0)\otimes\rho_B$, and we choose this $\rho_B$ in the P projector $\rightarrow P\rho(t_0)=\rho(t_0)\rightarrow Q\rho(t_0)=0$
- The initial correlations correspond to an inhomogeneous term in the time evolution.
- $\mathcal{L}$ act between the $P$ and $Q$ sectors $\rightarrow P\mathcal{L}(t)P\rho(t)=0$ (we may have to redefine the bath operators)

# $P\mathcal{L}(t)P\rho(t)$ details



$P\mathcal{L}P\rho = P\mathcal{L}\,\mathrm{Tr}_B(\rho)\otimes\rho_B = -iP[H_I, \rho_S\otimes\rho_B] =$

$= -i\sum_\alpha\left[\mathrm{Tr}_B(A_\alpha\rho_S\otimes B_\alpha\rho_B)\otimes\rho_B - \mathrm{Tr}_B(\rho_S A_\alpha\otimes\rho_B B_\alpha)\otimes\rho_B\right] =$

$= -i\sum_\alpha [A_\alpha,\rho_S]\langle B_\alpha\rangle\otimes\rho_B = 0,$ mert $B_\alpha$ a bath operátorok átdefiniálásával 0-vá tehető

Legyen $\rho_B(0) = \sum_\mu \lambda_\mu |\mu\rangle\langle\mu|$   $B_d(t) = \sum_\mu B_{\mu\mu}(t)|\mu\rangle\langle\mu|$ $\nwarrow$ $\rho_B$ s.v.

$B_{\mu\mu} = \langle\mu|B(t)|\mu\rangle$   $B'(t) = B(t) - B_d(t)$

$\langle B'(t)\rangle = \sum_\mu \lambda_\mu\langle\mu|B(t)|\mu\rangle - \sum_\mu\langle\mu|\left[\sum_\nu B_{\nu\nu}(t)|\nu\rangle\langle\nu|\right]|\mu\rangle = 0$

I'm not sure why this redefinition is allowed.
Somehow $[H_B, \rho_B(0)] = 0$ is needed.

# Born approximation

$$\frac{\partial}{\partial t} P\rho(t) = \int_{t_0}^{t} ds K(t,s) P\rho(s)$$

$$K(t,s) = \alpha^2 P\mathcal{L}(t) G(t,s) Q\mathcal{L}(s) P$$

$$G(t,s) = T_{\leftarrow} \exp\left[\alpha \int_{s}^{t} ds' Q\mathcal{L}(s')\right]$$

Second order approximation in $\alpha \rightarrow G(t,s) = 1$

$$\frac{\partial}{\partial t}\rho_s(t) = -\alpha^2 \int_{t_0}^{t} \mathrm{Tr}_B[H_I(t), [H_I(s), \rho_s(s) \otimes \rho_B]]$$

Now we see that this is actually second order in $\alpha$, but now we do not have to lie that it is because the bath is *reasonably large*, changes in the environment are *negligible* and the system and the bath remains in an approximate product state.
Back to the book.

# Time convolutionless projection

$$K_2(t) = P\mathcal{L}(t)\Sigma_1(t)P$$

$$\Sigma_1(t) = \int_{t_0}^{t} dt_1 Q\mathcal{L}(t)P$$

$$K_2(t) = \int_{t_0}^{t} dt_1 P\mathcal{L}(t)\mathcal{L}(t_1)P$$

$$\frac{\partial \rho_s(t)}{\partial t} = -\alpha^2 \int_{t_0}^{t} ds \mathrm{Tr}_B[H_I(t), [H_I(s), \rho_s(t) \otimes \rho_B]]$$

At the level of second order approximation we can simply change $\rho(s)$ to $\rho(t)$.

Question: does $(1 - \Sigma(t))^{-1}$ exist? What does it mean, if it does not?

$$\frac{\partial}{\partial t} P\rho(t) = I(t)Q\rho(t_0) + K(t)P\rho(t)$$

$$\begin{aligned} I(t) &= \alpha P\mathcal{L}(t)[1 - \Sigma(t)]^{-1} G(t, t_0)Q \\ K(t) &= \alpha P\mathcal{L}(t)[1 - \Sigma(t)]^{-1} P \end{aligned}$$

# Beyond the Kraus representation

# Kraus operators

$$\rho_{s\epsilon}(0) = \rho_s(0) \otimes \rho_\epsilon(0)$$

$$\begin{aligned}\rho_s(t) &= \mathrm{Tr}_\epsilon \left( U_{s\epsilon}\, \rho_{s\epsilon}(0)\, U_{s\epsilon}^{+} \right) = \sum_\mu \langle \mu | U_{s\epsilon}\, \rho_s(0) \otimes \sum_\nu p_\nu |\nu\rangle\langle\nu|\, U_{s\epsilon}^{+} |\mu\rangle \\ &= \sum_{\mu\nu} \langle \mu | \sqrt{p_\nu}\, U_{s\epsilon} |\nu\rangle\, \rho_s(0)\, \langle \nu | \sqrt{p_\nu}\, U_{s\epsilon}^{+} |\mu\rangle \\ &= \sum_{\mu\nu} M_{\mu\nu}\, \rho_s(0)\, M_{\mu\nu}^{+}\end{aligned}$$

$$M_{\mu\nu} = \langle \mu | \sqrt{p_\nu}\, U_{s\epsilon} |\nu\rangle$$

# Beyond the Kraus representation [2]

If we do not start from a product state:

$$\rho_{s\epsilon}(0) = \frac{1}{NM}\left(\mathbb{I}_{s\epsilon} + \alpha_i\,\sigma_i \otimes \mathbb{I}_\epsilon + \beta_j\,\mathbb{I}_s \otimes \tau_j + \gamma_{ij}\,\sigma_i \otimes \tau_j\right)$$

where $\sigma_i \quad i = 1, \dots, N^2 - 1 \quad S \quad \mathrm{SU}(N)$ generators

$\tau_j \quad j = 1, \dots, M^2 - 1 \quad \epsilon \quad \mathrm{SU}(M)$ generators

$$\rho_s(0) = \mathrm{Tr}_\epsilon\,\rho_{s\epsilon} = \frac{1}{N}\left(\mathbb{I}_s + \alpha_i\,\sigma_i\right)$$

# Beyond the Kraus...

$$\rho_s(t) = \sum_\mu \langle\mu| U_{s\epsilon} \left( \frac{1}{N}(\mathbb{I}_s + \alpha_i \sigma_i) \otimes \frac{1}{M}(\mathbb{I}_\epsilon + \beta_j \tau_j) \right) U_{s\epsilon}^{+} |\mu\rangle$$

$$+ \sum_\mu \langle\mu| U_{s\epsilon} \frac{\gamma_{ij} - \alpha_i \beta_j}{NM}\, \sigma_i \otimes \tau_j \, U_{s\epsilon}^{+} |\mu\rangle =$$

$$= \sum_\mu \langle\mu| U_{s\epsilon} (\rho_s \otimes \rho_\epsilon) U_{s\epsilon}^{+} |\mu\rangle \quad \text{(Kraus)}$$

$$+ \sum_\mu \langle\mu| U_{s\epsilon} \gamma'_{ij}\, \sigma_i \otimes \tau_j \, U_{s\epsilon}^{+} |\mu\rangle$$

$$\rho_s^{(1)}(0) = \rho_s^{(2)}(0) \qquad \rho_\epsilon^{(1)}(0) = \rho_\epsilon^{(2)}(0)$$

it does not follow that $\rho_s^{(1)}(t) = \rho_s^{(2)}(t)$

# Something seems to be linear

$$\rho_s(t) = T(t)\rho_s(0) + \xi(t)$$

$$\frac{\partial}{\partial t}\rho_s(t) = \frac{\partial}{\partial t}T(t)\rho_s(0) + \frac{\partial}{\partial t}\xi(t)$$

$$\rho_s(0) = T(t)^{-1}(\rho_s(t) - \xi(t))$$

$$\frac{\partial}{\partial t}(\rho_s(t) - \xi(t)) = \left(\frac{\partial}{\partial t}T\right) T^{-1}(t)(\rho_s(t) - \xi(t))$$

$$X = \left(\frac{\partial}{\partial t}T\right) T^{-1}(t)$$

$$\frac{\partial}{\partial t}(\rho_s(t) - \xi(t)) = X(\rho_s(t) - \xi(t))$$

$$\xi(t) = \mathrm{Tr}_\epsilon\left(U_{s\epsilon}(\rho_{s\epsilon}(0) - \rho_s(0)\otimes\rho_\epsilon(0))U_{s\epsilon}^{+}\right)$$

There is always something that evolves linearly, but it is not the $\rho_s(t)$. What is more, $\xi(t)$ is non-linear in $\rho_{s\epsilon}(0)$.

# Affine maps of density matrices [3]

For quantum systems, linear and affine maps of matrices are shown to provide equivalent descriptions of evolution of density matrices for a subsystem.

There may be significant advantage in using affine map. The linear map is generally not CP, but the linear part of the equivalent affine map can be chosen CP.

# Initial correlations and linear dynamical maps [4]

Without initial correlation: $\rho_s(t) = V(t)\rho_s(0)$.
With initial correlation: $\rho_{s\epsilon}(0) = \rho_s(0) \otimes \rho_\epsilon(0) + \xi(0)$,

$$V^\xi(t)\rho_s(0) = V(t)\rho_s(0) + I^\xi(t)$$

We would like to be able to substitute any $\rho_s(0)$ as initial condition. However this is not possible. For given $\xi(0)$ and $\rho_\epsilon(0)$, the allowed 'physical' states are:

$$\mathcal{P}^\xi_{\rho_\epsilon} = \{\rho_s \mid \rho_s \otimes \rho_\epsilon + \xi \geqslant 0\}$$

The $V^\xi(t)$ can 'absorb' the affine term in a 'linear way':

$$W(t)\rho_s(0) = V(t)\rho_s(0) + I^\xi \mathrm{Tr}\rho_s(0)$$

In this case the main advantage is that the generator of the evolution can be put in a generalized Lindblad form with time-dependent coefficients.

**Questions?**

# Some possible questions

- We got used to separating the initial conditions and the governing laws. Here it seems that there is an interplay between them.
- Invertibility of $T$ or $(1 - \Sigma)$.
- D. Schmid, K. Ried, R. W. Spekkens: Why initial system-environment correlations do not imply the failure of complete positivity: A causal perspective [5]

# Citations i

[1] Heinz-Peter Breuer and Francesco Petruccione.
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Oxford University Press, 01 2007.

[2] Peter Štelmachovič and Vladimír Bužek.
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[3] Thomas F. Jordan.
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*Phys. Rev. A*, 71:034101, Mar 2005.

[4] Alessandra Colla, Niklas Neubrand, and Heinz-Peter Breuer.
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[5] David Schmid, Katja Ried, and Robert W. Spekkens.
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