Resurgence II

Examples in quantum theory and alien calculus

Istvan Vona

December 17, 2025

Elte University, Wigner RCP

Outline

Quartic integral

Toy model for path integral, $g \ll 1$ expansion

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4} \sim \sum_{n=0}^{\infty} c_n g^n = 1 - \frac{3}{4} g + \frac{105}{32} g^2 - \frac{3465}{128} g^3 + O(g^4)$$

Asymptotic series

$$c_n = \frac{(-1)^n \Gamma\left(2n + 1/2\right)}{\Gamma(1/2)\Gamma(n+1)}, \qquad R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = 0$$

Same coefficients "resurge"

$$c_n \sim \frac{1}{\sqrt{2\pi}} \left\{ 1 \cdot \frac{\Gamma(n)}{A^n} + \frac{3}{4} \frac{\Gamma(n-1)}{A^{n-1}} + \frac{105}{32} \frac{\Gamma(n-2)}{A^{n-2}} + \frac{3465}{128} \frac{\Gamma(n-3)}{A^{n-3}} + \dots \right\}, \quad A = -\frac{1}{4}$$

Asymptotic series

Formal power series

$$\varphi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

asymptotic to a function $f(x)$

$$f(x) \sim \varphi(x), \quad \text{for} \quad x \to x_0 \quad \text{if} \quad \exists C_N : \quad \left| f(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \right| \leq C_N (x - x_0)^{N+1}$$

Not unique

$$e^{-1/x} = 0 + 0 \cdot x + 0 \cdot x^2 + \ldots \sim 0, \quad \text{for} \quad x \to 0$$

In physics: Gevrey-order 1

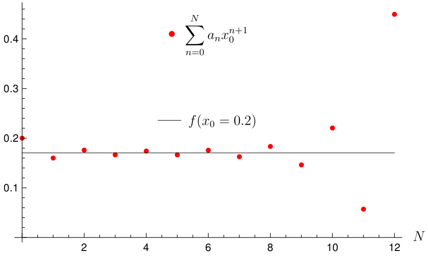
$$|a_n| \leq K c^n (n!)$$

Optimal truncation

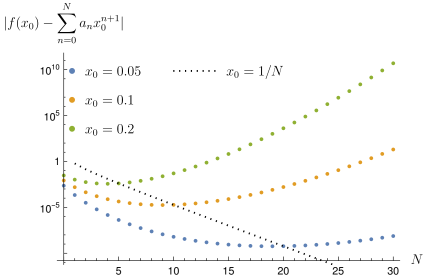
Truncation at finite order n if $x \ll 1 \Rightarrow$ polynomial error

Optimal point to truncate

$$\min_n (n! x^n) \quad \Rightarrow \quad n_{\text{opt}} \sim 1/x \quad \Rightarrow \quad \text{error} \propto e^{-1/x}$$



(a) Divergence



(b) Optimal truncation

Borel summation

"Divide out" factorial growth

$$\hat{\varphi}(t) \equiv \mathcal{B}[\varphi](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \qquad \Rightarrow \text{convergent inside } R = \text{finite}$$

Analytical continuation $\Rightarrow$ integration

$$\int_0^\infty \mathrm{d}t\, e^{-t/x} t^n = n! x^{n+1} \quad \text{(restores factorial)}$$

Laplace-integral

$$S(\varphi(x)) = \mathcal{L}[\hat{\varphi}](x) = x^{-1} \int_0^\infty \mathrm{d}t\, e^{-t/x} \hat{\varphi}(t) = \int_0^\infty \mathrm{d}t\, e^{-t} \hat{\varphi}(tx), \qquad S = \mathcal{L} \circ \mathcal{B}$$

Finite summability

$$\varphi(x) = a_0 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1}, \quad \mathcal{B}[\varphi](t) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} t^n, \quad S(\varphi(x)) = a_0 + \int_0^\infty \mathrm{d}t\, e^{-t/x} \mathcal{B}[\varphi](t)$$

Euler ODE

ODE

$$\mp f'(z) + f(z) = \frac{1}{z}$$

Homogeneous solutions

$$f_h(z) = ce^{\pm z}$$

Asymptotic expansion for $z \gg 1$

$$f(z) \sim \varphi(z) = \sum_{n=0}^{\infty} a_n z^{-n-1}, \qquad a_n = (\mp 1)^n n!$$

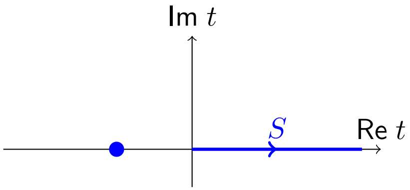
Borel-transform has a pole at $t_0 = \mp 1$

$$\mathcal{B}[\varphi](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n = \sum_{n=0}^{\infty} (\mp t)^n = \frac{1}{1 \pm t}$$

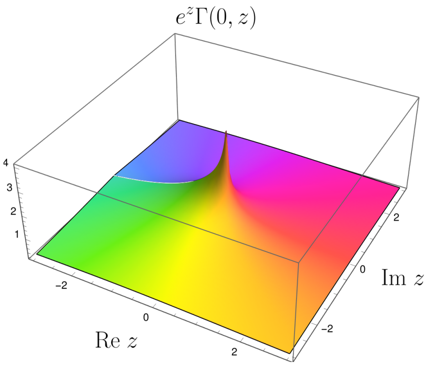
Alternating factorial

Borel summable

$$S(\varphi(z)) = \int_0^\infty \mathrm{d}t\, \frac{e^{-tz}}{1+t} = e^z\Gamma(0,z) \quad \Rightarrow \quad f(z) = \underbrace{S(\varphi(z))}_{\text{particular}} + \underbrace{ce^z}_{\text{homogeneous}}$$



(c) Borel-plane



(d) Incomplete gamma

Alternating factorial

Rotating the contours

$$\begin{cases} z \to e^{i\theta} z \\ t \to e^{-i\theta} t \end{cases} \quad : \quad S(\varphi(e^{i\pi}z)) - S(\varphi(e^{-i\pi}z)) = -2\pi i \text{Res}_{t=-1} \frac{e^{tz}}{1+t} = -2\pi i e^{-z}$$

Connection formula

$$e^{-z}\Gamma(0, e^{i\pi}z) - e^{-z}\Gamma(0, e^{-i\pi}z) = -2\pi i e^{-z}$$

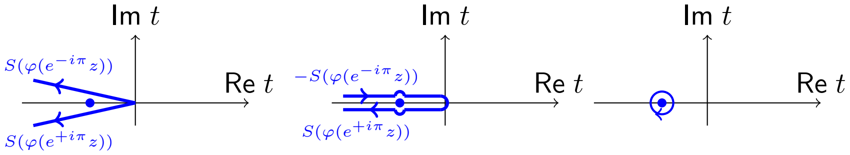
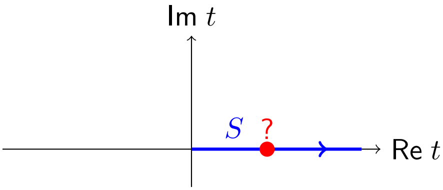
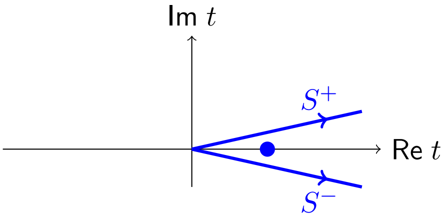


Figure 1: Rotated contours

Non-alternating factorial

Integration hits singularity on the Borel plane



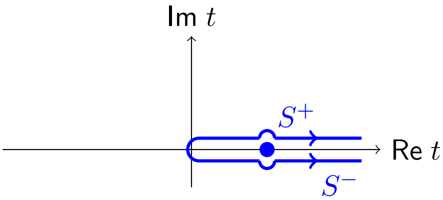


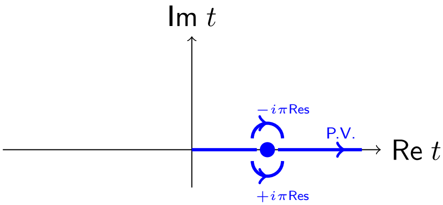
$$S^{\pm}(\varphi(z)) = \int_0^{\infty e^{\pm i0}} \mathrm{d}t \, \frac{e^{-tz}}{1-t}$$

Non-alternating factorial

Imaginary **ambiguities** appear

$$S^{\pm}(\varphi(z)) = \int_0^{\infty} \mathrm{d}t\, \frac{e^{-tz}}{1 \mp i0 - t} = \text{P.V.} \int_0^{\infty} \mathrm{d}t\, \frac{e^{-tz}}{1-t} \mp i\pi\, \text{Res}_{\,t=1}\, \frac{e^{-tz}}{1-t}$$





$$S^{\pm}(\varphi) = S(\varphi) \pm i\pi e^{-z}, \qquad S(\varphi) \equiv \frac{S^+(\varphi) + S^-(\varphi)}{2} = e^{-z}\text{Ei}(z)$$

Non-alternating factorial

Physical solution: lateral Borel resummation of a **(ambiguity-free) trans-series**

$$f(z) = \underbrace{S(\varphi)}_{\text{particular}} + \underbrace{ce^{-z}}_{\text{homogeneous}} = S^{\pm}(\varphi + (c \mp i\pi)e^{-z}) = S^{\pm}\underbrace{\left(\sum_{n=0}^{\infty} n! x^{n+1} + (c \mp \overbrace{i\pi}^{\text{Stokes}})e^{-z}\right)}_{\text{trans-series}}$$

"Ambiguity cancellation"

$$\mathrm{Im} f(z) = \cancel{\mathrm{Im}\, S^{\pm}(\varphi)} \mp \cancel{\pi e^{-z}} = 0$$

Outline

Read

Marcos Marino: An introduction to resurgence in quantum theory [1]

Perturbation theory

Bender & Wu [2] anharmonic oscillator ($m, \hbar = 1$):

$$-\frac{1}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x) + V(x)\psi(x) = E(g)\psi(x), \qquad V(x) = \frac{1}{2}\omega^2 x^2 + gx^4$$

Idea: harmonic oscillator ansatz for wavefunctions

$$\psi(x) = e^{-\frac{\omega}{2}x^2}\sum_{\ell=0}^{\infty} u_\ell(x) g^\ell, \qquad E = \sum_{\ell=0}^{\infty} \epsilon_\ell g^\ell$$

0^{th} order: Hermite polynomials

$$u_0(x): \qquad \epsilon_0 = \omega\left(\nu + \frac{1}{2}\right)$$

ℓ^{th} order:

$$u_\ell(x) = \sum_{k=0}^{K_\ell} A_\ell^k x^k, \qquad K_\ell \leq 6\ell$$

Perturbation theory

Recursion (for ground state, $\epsilon_0 = \frac{1}{2}\omega$):

$$A_\ell^0 = \delta_{\ell,0} \qquad A_\ell^k = \frac{1}{2\omega k}\left[A_\ell^{k+2}(k+2)(k+1) + 2\sum_{\ell'=1}^{\ell-1}\epsilon_{\ell'}A_{\ell-\ell'}^k - 2A_{\ell-1}^{k-4}\right],\ k>0$$

$$\epsilon_\ell = -A_\ell^2$$

Ground state energy

$$E_0(g) \sim \frac{1}{2} + \frac{3g}{4} - \frac{21g^2}{8} + \frac{333g^3}{16} - \frac{30885g^4}{128} + \frac{916731g^5}{256} + O\left(g^6\right)$$

For generic potentials, any energy level: BenderWu package [3]

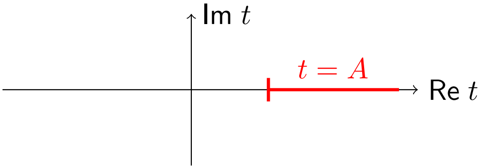
Extracting large order behaviour numerically

More generic singularity

$$f(x) \sim \varphi(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{A^n \Gamma(\alpha)} x^n \quad \Rightarrow \quad \hat{\varphi}(t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{A^n \Gamma(\alpha)\Gamma(n+1)} t^n = \left(1 - \frac{t}{A}\right)^{-\alpha}$$

Leading parameters

$$\xi_n \equiv \frac{n a_n}{a_{n+1}} = A - \frac{\alpha}{n} + O(n^{-2})$$

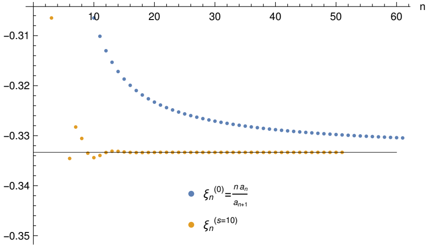


Extracting large order behaviour

Series acceleration (Richardson extrapolation)

$$\xi_n^{(s+1)} = \frac{1}{s+1}\Big((n-s)\xi_n^{(s)} - (n-1-2s)\xi_{n-1}^{(s)}\Big)\,, \quad \xi_n^{(0)} \equiv \xi_n$$

$$\xi_n = A + O(n^{-1}), \qquad\qquad \xi_n^{(s)} = A + O(n^{-s-1})$$



Turns out $A = -1/3, \alpha = 1/2$

Padé approximant

Approximating finite Borel transform with a rational function

$$\mathcal{B}_N[\varphi](t) = \sum_{n=0}^{N} \frac{a_n}{n!} t^n \quad \overset{!}{=} \quad \frac{p_0 + p_1 t + p_2 t^2 + \ldots + p_{N/2} t^{N/2}}{1 + q_1 t + q_2 t^2 + \ldots + q_{N/2} t^{N/2}} + O(t^{N+1})$$

Roots of denominator condense around branchcuts

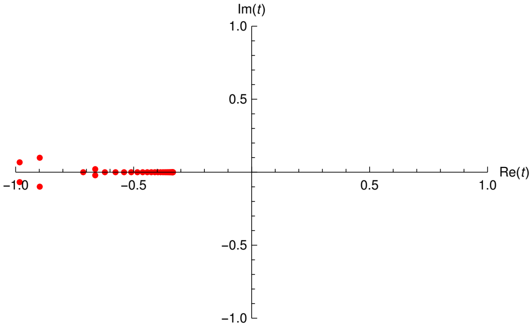


Figure 2: Roots of the Padé denominator

Leading behaviour

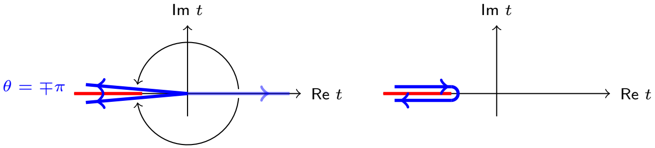
$$E_0(g) \sim \sum_{n=0}^{\infty} a_n g^n$$

Leading behaviour

$$a_n \sim -\frac{\sqrt{6}}{\pi^{3/2}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(-1/3)^n} \qquad \overset{A=-1/3,\quad \alpha=1/2}{\Longrightarrow} \qquad \hat{\varphi}(t) \sim -\frac{\sqrt{6}}{\pi} \frac{1}{\sqrt{1+3t}}$$

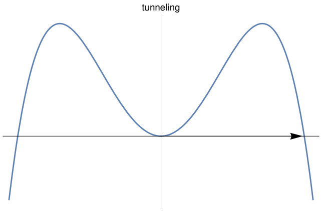
Discontinuity encodes exponentially small correction

$$\int_0^{\infty} \mathrm{d}t \; e^{-t/g} \text{Disc } \hat{\varphi}(-t) \sim \frac{1}{\sqrt{g}} e^{-\frac{1}{3|g|}} \propto \text{Im} E_0(-|g|)$$



Instanton effect

Potential for $g < 0$ is $V(x) = \frac{1}{2}x^2 - |g|x^4$



Instanton

$$\frac{1}{2}\dot{x}^2 \underbrace{-\frac{1}{2}x^2 + |g|x^4}_{\text{inverted potential}} = E \quad \Rightarrow \quad x_{\mathsf{c}}(t) = \sqrt{\frac{2}{|g|}}\frac{1}{\cosh(t - t_0)}$$

Instanton effect

The classical action is

$$\mathcal{S}_c = \mathcal{S}[x_c(t)] = \frac{1}{3|g|}$$

Gives non-perturbative contribution in the path integral

$$\int \mathcal{D}x \; e^{-\mathcal{S}[x]} \sim \text{perturbative} + e^{-\mathcal{S}_c}(\ldots)$$

Infinitely many such multi-instanton contributions!

$$t = -\frac{\ell}{3}, \quad \ell = 1, 2, 3, \ldots$$

Reconstructing the exact function

This is Borel summable for $g > 0$!

Using Padé-Borel (see also Section 4 of [4] and papers by O. Costin & G. Dunne)

$$\mathcal{PB}[\varphi](t) = \frac{p_0 + p_1 t + p_2 t^2 + \ldots + p_{N/2} t^{N/2}}{1 + q_1 t + q_2 t^2 + \ldots + q_{N/2} t^{N/2}}$$

We can integrate

$$S(\varphi) = \int_0^\infty \mathrm{d}t \; e^{-t/g} \mathcal{PB}[\varphi](t)$$

Typically gives better result than optimal truncation! For $N = 180$ precise up to ~ 21 digits

$$E_0(g=1) = S(\varphi)|_{g=1} = 0.80377065123427376935 4..$$

Outline

Read

D. Dorigoni: An Introduction to Resurgence, Trans-Series and Alien Calculus
1411.3585 [5]

Basics

Borel transform

$$\mathcal{B}\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) \equiv \sum_{n=0}^{\infty} \frac{a_n}{n!} s^n, \qquad \phi(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \Rightarrow \quad \hat{\phi}(s) \equiv \mathcal{B}\left(\phi(x)\right)$$

We will use

$$x \to 0 \qquad \Leftarrow \qquad z = 1/x, \qquad \Rightarrow \qquad z \to \infty$$

Borel-transform **multiplicative model** $\to$ **convolutional model**

$$\mathcal{B}\left(\phi_1 \phi_2\right) = \mathcal{B}\left(\phi_1\right) * \mathcal{B}\left(\phi_2\right) = \int_0^s \mathrm{d}t \; \hat{\phi}_1(s-t)\hat{\phi}_2(t)$$

Properties

$$z^{-\alpha} \xrightarrow{\mathcal{B}} \frac{s^{\alpha-1}}{\Gamma(\alpha)} \qquad\qquad \partial_z \phi(z) \xrightarrow{\mathcal{B}} -s\hat{\phi}(s)$$

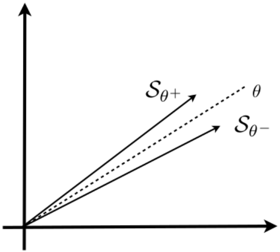
Basics

Directional laplace transform

$$\mathcal{L}_\theta(\hat{\phi}) \equiv \int_0^{\infty e^{i\theta}} \mathrm{d}s \; e^{-s/x} \hat{\phi}(s), \qquad \mathcal{L} \equiv \mathcal{L}_{\theta=0}$$

Lateral Borel resummations around a direction

$$S_\theta^\pm(\phi) \equiv \mathcal{L}_{\theta\pm 0}(\mathcal{B}(\phi)), \quad \theta \in [0, 2\pi], \qquad S^\pm \equiv S^\pm_{\theta=0}$$



Trans-series

Encodes non-perturbative corrections of an asymptotic expansion

$$f(x) \sim \phi_0(x) + \phi_1(x)e^{-\omega_1/x} + \phi_2(x)e^{-\omega_2/x} + \ldots = \sum_{k=0}^{\infty} \phi_k(x)e^{-\omega_k/x}$$

Double series in $x, \quad e^{-1/x}$

$$\omega_k = k : \quad \Rightarrow \quad f(x) \sim \sum_{k,n=0}^{\infty} \phi_{k,n} x^n e^{-k/x}$$

Could involve also logarithm

$$x, \qquad\qquad e^{-1/x}, \qquad\qquad \ln x$$

“Weak” resurgence

Physical quantities

$$f(x) = S^{\pm}\,(\text{ambiguity free trans-series}) = \sum_{k=0}^{\infty} S^{\pm}\left(\phi_k^{\pm}(x)\right) e^{-\omega_k/x}$$

Also called **Borel-Écalle resummation**

Cancellation of ambiguities among imaginary parts

$$\sum_{k=0}^{\infty} \mathrm{Im} S^{\pm}\left(\phi_k^{\pm}(x)\right) e^{-\omega_k/x} = 0$$

Properties of $S^{\pm}$

Linearity

$$S_\theta^{\pm}(a\phi_1 + b\phi_2) = aS_\theta^{\pm}(\phi_1) + bS_\theta^{\pm}(\phi_2)$$

Multiplication

$$S_\theta^{\pm}\left(\phi_1\phi_2\right) = S_\theta^{\pm}\left(\phi_1\right) S_\theta^{\pm}\left(\phi_2\right)$$

Composition with analytic function

$$S_\theta^{\pm}\left(f \circ \phi\right) = f(S_\theta^{\pm}(\phi))$$

Properties of $S^{\pm}$

Real coefficients

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad a_n \in \mathbb{R}$$

For real series $S^{\pm}$ are conjugate pairs

$$\left(S^{+}(\phi)\right)^{*} = S^{-}(\phi) \quad \text{if} \quad x \in \mathbb{R}$$

Define their "mean"

$$\Phi(x) \equiv \frac{S^{+}(\phi) + S^{-}(\phi)}{2} \quad \Rightarrow [\Phi(x)]^{*} = \Phi(x^{*}) \quad x \in \mathbb{C}$$

Properties of $S^{\pm}$

Non-alternating series

$$\phi(x) = \sum_{n=0}^{\infty} n! x^{n+1}$$

Principal Value = real part

$$S^{+}(\phi) = \mathrm{Re}\, S^{+}(\phi) + i\pi e^{-1/x}, \quad x \in \mathbb{R}$$

Multiplication property

$$S^{+}\left(\phi^{2}\right) = \left[S^{+}\left(\phi\right)\right]^{2} = \left[\mathrm{Re}\, S^{+}(\phi)\right]^{2} + i\pi e^{-1/x}\mathrm{Re}\, S^{+}(\phi(x)) - \pi^{2} e^{-2/x}$$

Real part fails to respect multiplication!

$$\mathrm{Re}\, S^{+}\left(\phi^{2}\right) = \left[\mathrm{Re}\, S^{+}(\phi)\right]^{2} \color{red}{-\pi^{2} e^{-2/x}}$$

Resurgent functions

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

Subleading factorial corrections, A, A_k: **asymptotic coefficients**

$$a_n \sim An! + A_0(n-1)! + A_1(n-2)! + A_2(n-3)! + \ldots A_k(n-k)! + \ldots$$

or

$$a_n \sim n! \left(A + \frac{A_0}{n} + \frac{A_1}{n(n-1)} + \frac{A_2}{n(n-1)(n-2)} + \ldots + \frac{A_k}{n(n-1)(n-2)\ldots(n-k)} + \ldots \right)$$

Borel transform $\rightarrow$ logarithmic cut

$$\hat{\phi}(s) \sim \frac{A}{1-s} - \left(A_0 + A_1(s-1) + \frac{1}{2}A_2(s-1)^2 + \ldots + \frac{A_k}{k!}(s-1)^2 + \ldots \right) \ln(1-s),$$

Resurgent functions

Function multiplying the cut:

$$\hat{\phi}(s) \sim \frac{A}{1-s} - \hat{\varphi}(s-1)\ln(1-s) + \mathsf{reg}(s-1), \qquad \hat{\varphi}(s) \equiv \sum_{k=0}^{K} \frac{A_k}{k!} s^k, \tag{1}$$

Discontinuity of the logarithm and pole at $s = \omega$

$$\frac{1}{\omega - (s \pm i0)} = \text{P.V.}\frac{1}{\omega - s} \pm i\pi\delta(\omega - s), \quad \ln(\omega - (s \pm i0)) = \begin{cases} \ln(\omega - s) & s < \omega \\ \ln|\omega - s| \mp i\pi & \omega \leq s \end{cases},$$

Discontinuity

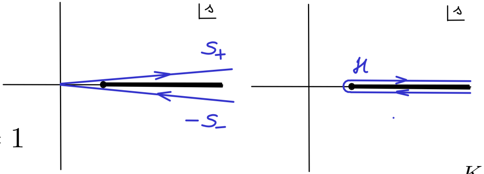
$$\text{Disc}\hat{\phi}(s) \equiv \frac{1}{2\pi i}\left(\hat{\phi}(s+i0) - \hat{\phi}(s-i0)\right) = A\delta(s-1) + \Theta(s-1)\hat{\varphi}(s-1),$$

Resurgent functions

Difference of lateral resummations

$$\left(S^{+}-S^{-}\right)\phi(x)=2\pi i\left(Ae^{-1/x}+e^{-1/x}\int_{1}^{\infty}\mathrm{d}s\ e^{-(s-1)/x}\hat{\varphi}(s-1)\right)$$

$$=2\pi i e^{-1/x}\mathcal{L}\big(\underbrace{A\delta(s)+\hat{\varphi}(s)}_{\Delta_1\hat{\phi}(s)}\big)=e^{-1/x}\Delta_1\phi(x),$$



Alien derivative at $s=1$

$$\Delta_1\phi(x)=2\pi i\left(A+A_0x+A_1x^2+\ldots\right)=2\pi i\sum_{k=-1}^{K}A_kx^{k+1}$$

Multiplying ambiguities

In general

$$A_k \propto k!$$

Their asymptotic looks

$$A_k \sim \tilde{A}k! + \tilde{A}_0(k-1)! + \tilde{A}_1(k-2)! + \tilde{A}_2(k-3)! + \dots \quad \text{for} \quad k \to \infty$$

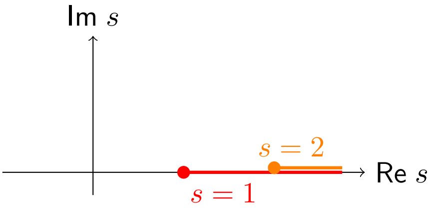
Borel transform

$$\hat{\phi}(s) \sim \frac{A}{1-s} - \underbrace{\left[\frac{\tilde{A}}{2-s} - \hat{\varphi}_2(s-2)\ln(2-s)\right]}_{\hat{\varphi}_1(s-1)} \ln(1-s)$$

where

$$\hat{\varphi}_2(s) \equiv \tilde{A}_0 + \tilde{A}_1 s + \frac{1}{2}\tilde{A}_2 s^2 + \dots + \frac{\tilde{A}_k}{k!}s^2 + \dots$$

Multiplying ambiguities



Discontinuities influence each other

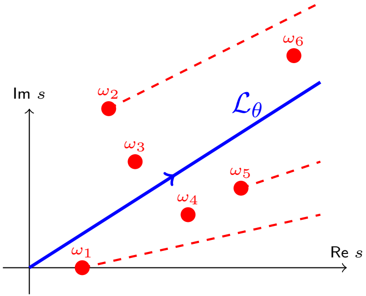
$$\ln(2-s)\ln(1-s) = \left(\ln|1-s| \mp i\pi\right)\left(\ln|2-s| \mp i\pi\right), \quad s>2$$

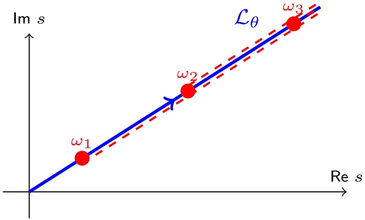
It gets complicated!

$$\text{Disc } \hat{\phi}(s) = A\delta(1-s) + \Theta(s-1)\left[\text{P.V.}\frac{\tilde{A}}{2-s} - \hat{\varphi}_2(s-2)\ln|2-s|\right] +$$
$$-\left[\tilde{A}\delta(2-s) + \Theta(s-2)\hat{\varphi}_2(s-2)\right]\ln|1-s|.$$

Alien calculus

Endless continuability





Convolution

Space of singularities closed under convolution

$$\hat{\phi}(s) = \frac{1}{\omega - s} \qquad \Rightarrow \qquad (1 * \hat{\phi})(s) = \int_0^s \mathrm{d}t \; \hat{\phi}(t) \sim -\ln(\omega - s)$$

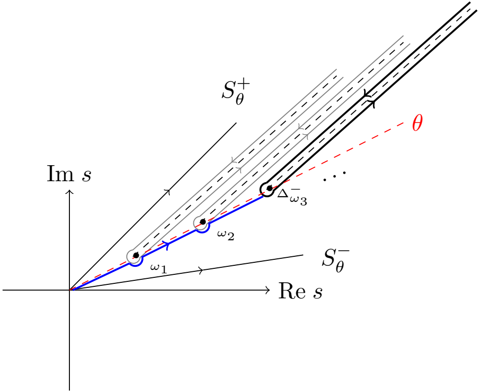
Convolving singularities

$$\hat{\phi}_1(s) = \frac{1}{\omega_1 - s}, \qquad \hat{\phi}_2(s) = \frac{1}{\omega_2 - s}$$

leads to new singularities

$$(\hat{\phi}_1 * \hat{\phi}_2)(s) = \frac{1}{\omega_1 + \omega_2 - s} \int_0^s \mathrm{d}t \left(\frac{1}{\omega_1 - t} + \frac{1}{\omega_2 - t} \right) \sim \frac{-\ln(\omega_1 - s) - \ln(\omega_2 - s)}{\omega_1 + \omega_2 - s}$$

Separating singularities



Picks up discontinuity after analytic continuation along the path γ_ω

$$\Delta^-_\omega \hat{\phi}(s) = \underbrace{A_{\gamma_\omega}}_{\text{residue}} \delta(s) + \underbrace{\hat{\varphi}_{\gamma_\omega}(s)}_{\text{minor}}.$$

Stokes automorphism

Transforming $S_\theta^- \to S_\theta^+$

$$\mathcal{L}_\theta^-(\hat{\phi}) = \mathcal{L}_\theta^+(\hat{\phi}) + e^{-\omega_1/x}\mathcal{L}_\theta^+\left(\Delta_{\omega_1}^-\hat{\phi}\right) + e^{-\omega_2/x}\mathcal{L}_\theta^+\left(\Delta_{\omega_2}^-\hat{\phi}\right) + \ldots,$$

Set of singularities

$$\Omega_\theta = \{\omega_1, \omega_2, \omega_3, \ldots\}, \qquad |\omega_1| < |\omega_2| < |\omega_3| < \ldots$$

Difference between lateral resummations: **Stokes automorphism**

$$S_\theta^- = S_\theta^+ \circ \underbrace{\left(\mathbb{1} + \sum_{\omega\in\Omega_\theta} e^{-\omega/x}\Delta_\omega^-\right)}_{\mathfrak{S}_\theta}$$

Properties of $\mathfrak{S}_\theta$

Relates $S_\theta^\pm$

$$S_\theta^- = S_\theta^+ \circ \mathfrak{S}_\theta, \qquad S_\theta^- \circ \mathfrak{S}_\theta^{-1} = S_\theta^+$$

Acts on trans-series!

$$\mathfrak{S}_\theta \phi = \phi + e^{-\omega_1/x} \Delta_{\omega_1}^- \phi + e^{-\omega_1/x} \Delta_{\omega_2}^- \phi + \dots$$

Preserves multiplication $\rightarrow$ **automorphism**

$$\mathfrak{S}_\theta \left(\phi_1 \phi_2\right) = \mathfrak{S}_\theta \left(\phi_1\right) \mathfrak{S}_\theta \left(\phi_2\right)$$

Its logarithm

$$D_\theta \equiv \log \mathfrak{S}_\theta \qquad \Rightarrow \qquad \mathfrak{S}_\theta^\alpha = \exp\left(\alpha D_\theta\right)$$

is a **derivation!**

$$D_\theta(\phi_1 \phi_2) = D_\theta(\phi_1)\phi_2 + \phi_1 D_\theta(\phi_2)$$

Alien derivatives

Introduce Δ_ω alien derivatives

$$\mathfrak{S}_\theta = \exp\left(D_\theta\right) = \exp\left(\sum_{\omega\in\Omega_\theta} e^{-\omega/x}\Delta_\omega\right)$$

They satisfy Leibniz rule

$$\Delta_\omega(\phi_1\phi_2) = \Delta_\omega(\phi_1)\phi_2 + \phi_1\Delta_\omega(\phi_2)$$

We may require

$$\log\mathfrak{S}_\theta = \log\left(\mathbb{1} + \sum_{\omega\in\Omega_\theta} e^{-\omega/x}\Delta^-_\omega\right) \stackrel{!}{=} \sum_{\omega\in\Omega_\theta} e^{-\omega/x}\Delta_\omega = D_\theta$$

Relation of Δ_ω and Δ^-_ω

Order by order in $e^{-\omega_i/x}$ we compare

$$
\begin{aligned}
\Delta_{\omega_1} &= \Delta^-_{\omega_1} \\
\Delta_{\omega_2} &= \Delta^-_{\omega_2} - \frac{1}{2}\left(\Delta^-_{\omega_1}\right)^2 \\
\Delta_{\omega_3} &= \Delta^-_{\omega_3} - \frac{1}{2}\left(\Delta^-_{\omega_1}\Delta^-_{\omega_2} + \Delta^-_{\omega_2}\Delta^-_{\omega_1}\right) + \frac{1}{3}\left(\Delta^-_{\omega_1}\right)^3 \\
&\vdots
\end{aligned}
$$

Explicit formula

$$\Delta_\omega = \sum_n \sum_{\omega_1+\omega_2+\ldots+\omega_n=\omega} \frac{(-1)^{n-1}}{n} \Delta^-_{\omega_1}\Delta^-_{\omega_2}\ldots\Delta^-_{\omega_n}$$

The Δ^-_ω fails Leibniz rule

$$\Delta^-_{\omega_2}(\phi_1\phi_2) = \Delta^-_{\omega_2}(\phi_1)\,\phi_2 + \Delta^-_{\omega_1}(\phi_1)\,\Delta^-_{\omega_1}(\phi_2) + \phi_1\Delta^-_{\omega_2}(\phi_2)$$

Exercise

Convolution of two poles

$$\hat{\phi}_1(s) = \frac{1}{\omega_1 - s}, \qquad \hat{\phi}_2(s) = \frac{1}{\omega_2 - s}, \qquad \hat{\phi}(s) = (\hat{\phi}_1 * \hat{\phi}_2)(s)$$

Calculate for each

$$\Delta_{\omega_1}, \quad \Delta_{\omega_2}, \quad \Delta_{\omega_1+\omega_2}, \quad \Delta_{\omega_1}\Delta_{\omega_2}, \quad \Delta_{\omega_2}\Delta_{\omega_2}$$

It turns out

$$\Delta_{\omega_1+\omega_2}\left(\hat{\phi}_1 * \hat{\phi}_2\right) = 0$$

despite

$$\hat{\phi}(s) \sim \frac{-\ln(\omega_1 - s) - \ln(\omega_2 - s)}{\omega_1 + \omega_2 - s}$$

Median resummation

For lateral resummations around $\theta = 0$

$$\left[S^{+}(\phi)\right]^{*} = S^{-}(\phi^{*}) \qquad \text{or} \qquad C \circ S^{+} = S^{-} \circ C$$

Relationship with Stokes automorphism

$$\mathfrak{S}^{-1} \circ C = C \circ \mathfrak{S} \qquad \Rightarrow \qquad -\Delta_\omega \circ C = C \circ \Delta_\omega$$

Alien derivative is imaginary

$$(\Delta_\omega \phi)^{*} = -\Delta_\omega \phi \quad \in i\mathbb{R}$$

[Remember

$$\Delta_1 \phi(x) = 2\pi i \left(A + A_0 x + A_1 x^2 + \ldots\right)]$$

Define

$$S^{\alpha} = S^{+} \circ \mathfrak{S}^{\alpha} \qquad \Rightarrow \qquad C \circ S^{\alpha} = S^{1-\alpha} \circ C$$

For $\alpha = 1/2$ we have something real!

$$\left[S^{1/2}(\phi)\right]^{*} = S^{1/2}(\phi^{*})$$

Median resummation

$$S^{\text{med}}(\phi) = S^{+}\left(\mathfrak{S}^{1/2}\phi\right) = S^{-}\left(\mathfrak{S}^{-1/2}\phi\right)$$

Gives real result for real coefficients and respects multiplication!

$$S^{\text{med}}(\phi_1\phi_2) = S^{\text{med}}(\phi_1)S^{\text{med}}(\phi_2)$$

Bridge equation

Ricatti-type **non-linear** equation

$$\partial_z F(z) + F(z) - F^2(z) = z^{-1}.$$

We introduce a trans-series parameter c

$$F(z) \sim \phi(z;c) = \sum_{k=0}^{\infty} c^k \phi_k(z) e^{-kz}$$

Normal and alien derivatives commute

$$[e^{-kz}\Delta_k, \partial_c] = 0$$

Simple way to calculate Δ_k: **Écalle's bridge equation** (see e.g. Dorigoni [5])

$$e^{-kz}\Delta_k \phi(z;c) \propto \partial_c \phi(z;c)$$

Strong resurgence

In particular

$$\Delta_1 \phi_k(z) = - \overbrace{2\pi i}^{\text{Stokes constant}} \cdot (k+1)\phi_{k+1}(z)$$

Stokes automorphism shifts the trans-series parameter

$$\mathfrak{S}^{\alpha}\phi(z;c) \sim \sum_{k=0}^{\infty} e^{-kz} \left(c - 2\pi i \alpha\right)^k \phi_k(z)$$

Median resummation

$$S^{\text{med}}(\phi(z;c)) = S^{+}\left(\mathfrak{S}^{1/2}\phi(z;c)\right) = \sum_{k=0}^{\infty} e^{-kz} \left(c - i\pi\right)^k S^{+}\left(\phi_k(z)\right)$$

If $c = 0$, using only the perturbative part we get the exact result!

$$\boxed{f(z) = S^{\text{med}}(\phi_0(z))}$$

Thank you!

Literature i

Lecture notes:

- Marcos Mariño: An introduction to resurgence in quantum theory [1]
- Marco Serone: Lectures on Resurgence in Integrable Field Theories [6]
- Gerald Dunne: Introductory Lectures on Resurgence [4]
- Daniele Dorigoni: An Introduction to Resurgence, Trans-Series and Alien Calculus [5]
- Brent Pym: Resurgence in geometry and physics [7]

Review papers:

- M. Serone: The Power of Perturbation Theory [8]

Literature ii

- I. Aniceto, G. Basar, R. Schiappa: A primer on resurgent transseries and their asymptotics [9]
- David Sauzin: Resurgent functions and splitting problems [10], Introduction to 1-summability and resurgence [11]

Theses:

- T. Reis: On the resurgence of renormalons in integrable theories [12]
- L. Schepers: Resurgence in deformed integrable models [13]

Books:

- R.B. Dingle: Asymptotic expansions: their derivation and interpretation [14]

Literature iii

- Jean Écalle: Les fonctions résurgentes (Vol. 1,2,3) [15], Guided tour through resurgence theory (short notes)[16]

- M. Marino, “An introduction to resurgence in quantum theory,” *Lecture notes available at https://www.marcosmarino.net* (2023) . `https://www.marcosmarino.net/uploads/1/3/3/5/133535336/resurgence-course.pdf`.
- C. M. Bender and T. T. Wu, “Anharmonic oscillator. II. a study of perturbation theory in large order,” *Physical Review D* **7** no. 6, (1973) 1620.
- T. Sulejmanpasic and M. Ünsal, “Aspects of perturbation theory in quantum mechanics: The BenderWu Mathematica ® package,” *Comput. Phys. Commun.* **228** (2018) 273–289, `arXiv:1608.08256 [hep-th]`.
- G. V. Dunne, “Introductory Lectures on Resurgence: CERN Summer School 2024,” `arXiv:2511.15528 [hep-th]`.

References ii

- D. Dorigoni, "An Introduction to Resurgence, Trans-Series and Alien Calculus," *Annals Phys.* **409** (2019) 167914, arXiv:1411.3585 [hep-th].
- M. Serone, "Lectures on Resurgence in Integrable Field Theories," *arXiv* (5, 2024) , arXiv:2405.02224 [hep-th].
- B. Pym, "Resurgence in geometry and physics." 2016. https://www.math.mcgill.ca/bpym/courses/resurgence/.
- M. Serone, G. Spada, and G. Villadoro, "The Power of Perturbation Theory," *JHEP* **05** (2017) 056, arXiv:1702.04148 [hep-th].

- I. Aniceto, G. Basar, and R. Schiappa, "A Primer on Resurgent Transseries and Their Asymptotics," *Phys. Rept.* **809** (2019) 1–135, `arXiv:1802.10441 [hep-th]`.
- D. Sauzin, "Resurgent functions and splitting problems," *arXiv* (2007) , `arXiv:0706.0137 [math.DS]`.
- D. Sauzin, "Introduction to 1-summability and resurgence," *arXiv* (5, 2014) , `arXiv:1405.0356 [math.DS]`.
- T. Reis, *On the resurgence of renormalons in integrable theories.* **PhD thesis, U. Geneva (main), 2022.** `arXiv:2209.15386 [hep-th]`.

- L. Schepers, "Resurgence in deformed integrable models." 2023.
  `https://cronfa.swan.ac.uk/Record/cronfa63254/Description`.
- R. B. Dingle, *Asymptotic expansions: their derivation and interpretation*.
  Academic Press, 1973.
- J. Écalle, *Les fonctions résurgentes (en trois parties)*.
  Université de Paris-Sud, 1981.
  Vol 1, 2, 3.
- J. Ecalle, "Guided tour through resurgence theory," *Jean Ecalle, Paris-Saclay* **435** (2022) 436–445. (link).

Outline

Stokes phenomenon

Airy function, linear potential in QM

$$\psi''(x) = x\psi(x) \quad \Rightarrow \quad -k^2\tilde{\psi}(k) = i\tilde{\psi}'(k) \quad \Rightarrow \quad \mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{i\left(\frac{k^3}{3}+kx\right)}$$

Asymptotic series as solution for $x \to \infty$

$$z \equiv \frac{4}{3}x^{3/2}, \qquad \psi_{\pm}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{\pm\frac{z}{2}}\varphi(\pm z), \qquad \varphi(z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

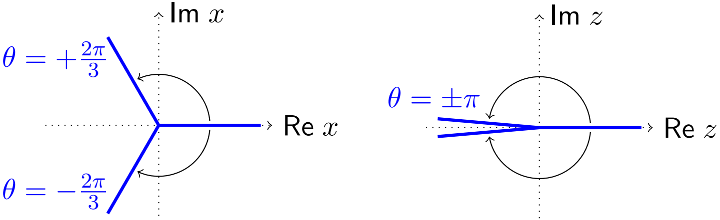
Two independent solutions to Airy ODE

$$\mathrm{Ai}(x) \sim \psi_-(x), \qquad \frac{1}{2}\mathrm{Bi}(x) \sim \psi_+(x)$$

Stokes phenomenon

Rotating

$$x \to e^{\pm 2\pi i/3}x, \qquad z \to -z \qquad \Rightarrow \qquad \psi_-(e^{\pm 2\pi i/3}x) = e^{\mp i\pi/6}\psi_+(x)$$



Naive expectation fails ↯

$$\mathrm{Ai}(e^{\pm 2\pi i/3}x) = \frac{1}{2}e^{\mp i\pi/6}\mathrm{Bi}(x) + \overbrace{\frac{i}{2}e^{\pm i\pi/3}\mathrm{Ai}(x)}^{\propto e^{-z}}$$

Stokes phenomenon

Borel-transform

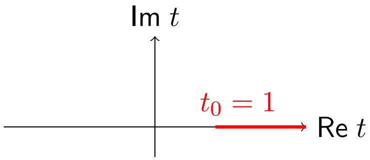
$$a_n = \frac{\Gamma(n+1/6)\Gamma(n+5/6)}{2\pi\Gamma(n+1)} = \{1, \frac{5}{72}, \frac{385}{10368}, \ldots\} \quad \Rightarrow \quad \hat{\varphi}(t) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; t\right)$$

Resurgent structure

$$a_n = \frac{1}{2\pi}\left(a_0\Gamma(n) - a_1\Gamma(n-1) + a_2\Gamma(n-2) - a_3\Gamma(n-3) + \ldots\right)$$

LO

$$a_n \sim \frac{\Gamma(n)}{2\pi} \quad \Rightarrow \quad \hat{\varphi}(t) \sim \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\Gamma(n)}{\Gamma(n+1)}t^n = -\frac{1}{2\pi}\ln(1-t)$$



Stokes phenomenon

Subleading terms

$$\sum_{\color{red}{n=k+1}}^{\infty} \frac{\Gamma(n-k)}{\Gamma(n+1)} t^n \sim \sum_{n=k+1}^{\infty} \frac{1}{n(n-1)\dots(n-k)} t^n$$

Differentiating k times

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} / \cdot \quad (\quad \dots \quad) = \sum_{n=k+1}^{\infty} \frac{1}{(n-k)} t^{n-k} = -\ln(1-t)$$

Singularity structure

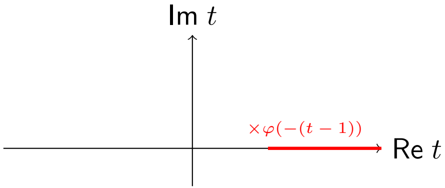
$$\overbrace{\int \mathrm{d}t \ \dots \int \mathrm{d}t}^{k} \left[-\ln\left(1-t\right)\right] = -\frac{1}{k!} \left(t-1\right)^k \ln\left(1-t\right) + \text{regular.}$$

Summing up

$$\varphi(t) = -\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k a_k}{k!}(t-1)^k \ln(1-t) + \text{regular}.$$

Function multiplying the cut is the same!

$$\varphi(t) = -\frac{1}{2\pi}\varphi(1-t)\ln(1-t) + \text{regular}.$$



Stokes-phenomenon

New notation

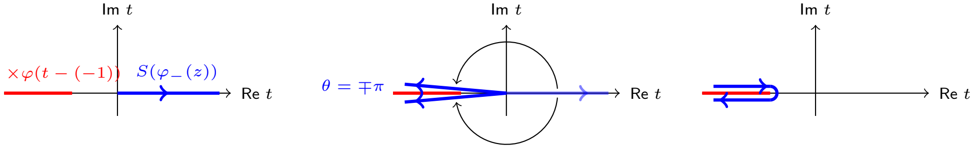
$$\mathrm{Ai}(x) \leftrightarrow \varphi(-z): \qquad \varphi_-(z) = \varphi(-z), \qquad \hat{\varphi}_-(t) = \hat{\varphi}(-t)$$

Borel summable

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{z(x)}{2}} S(\varphi_-(z)), \quad x > 0$$

Connection formula

$$S(\varphi_-(e^{i\pi}z)) - S(\varphi_-(e^{-i\pi}z)) = ie^{-z}S(\varphi_-(z))$$



Stokes phenomenon

Large $r \to \infty$ asymptotics of Airy

$$x = re^{i\theta}: \quad \mathsf{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{i\left(\frac{k^3}{3}+kx\right)} \overset{k=-iq\sqrt{r}}{=} \frac{\sqrt{r}}{2\pi i}\int_{-i\infty}^{i\infty} \mathrm{d}q\, e^{-r^{3/2}\mathcal{S}(q)}$$

$$\mathcal{S}(q) = \frac{q^3}{3} - qe^{i\theta}$$

Stationary points

$$\mathcal{S}'(q^*) = 0 \qquad \Rightarrow \qquad q^* = \pm e^{i\theta/2}$$

Steepest descent flow

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = -\overline{\mathcal{S}'(q)}, \qquad \mathsf{Im}\,\mathcal{S}(q) = \text{const.}, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\mathsf{Re}\,\mathcal{S}(q) < 0$$

Flow equations $q = u + iv$

$$\dot{u} = -u^2 + v^2 + \cos\theta, \qquad \dot{v} = 2uv - \sin\theta.$$

Stokes phenomenon

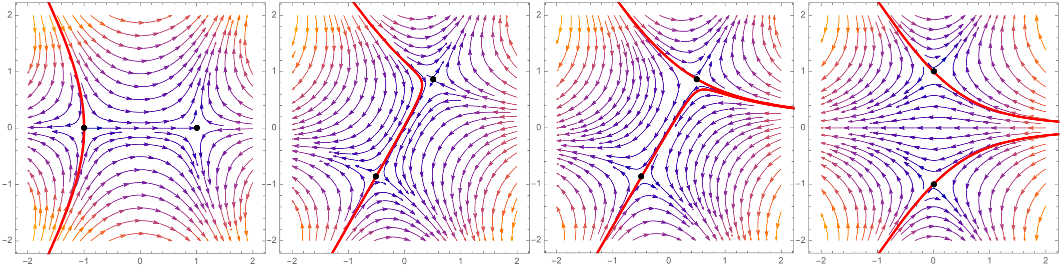


Figure 3: Flow for $\theta = 0,\ 2\pi/3 - \epsilon,\ 2\pi/3 + \epsilon,\ \pi$ respectively. [4]

Stokes phenomenon

$$\theta = 0 : \qquad \mathsf{Ai}(|x|) \sim \frac{1}{2\sqrt{\pi}|x|^{1/4}} e^{-\frac{2}{3}|x|^{3/2}}$$

$$\theta = \frac{2\pi}{3} : \quad \mathsf{Ai}(e^{2\pi i/3}|x|) = \frac{1}{2\sqrt{\pi}|x|^{1/4}} \left(e^{\frac{2}{3}|x|^{3/2} - i\pi/6} + \frac{i}{2} e^{-\frac{2}{3}|x|^{3/2} + i\pi/3} \right)$$

$$\theta = \pi : \qquad \mathsf{Ai}(-|x|) \sim \frac{1}{2\sqrt{\pi}|x|^{1/4}} \left(e^{-i\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)} + e^{i\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)} \right) \sim \frac{\cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)}{\sqrt{\pi}|x|^{1/4}}$$