Resurgence theory I

December 17, 2025

History

- Dyson [1] $\rightarrow$ perturbative series in QFT is asymptotic
- 19th century mathematicians developed methods to deal with such divergent series (e.g. Stokes, Poincaré, Stieltjes, Borel [2, 3, 4, 5, 6])
- Robert B. Dingle [7] $\rightarrow$ universality of factorial over power growth

$$a_n \sim \frac{n!}{A^n}$$

- Lipatov [8] suggested that instantons are behind the factorial growth of the coefficients, such that the $O(1)$ contribution of factorially many Feynman diagrams [9] sums up
- Lautrup [10] showed that there are also individual diagrams (renormalons) whose contribution grows as $n!$ with loop order n
- 't Hooft & Parisi [11],[12, 13]: the renormalons lead to different asymptotics than the instantons, the series is not always Borel summable
- In Quantum Mechanics, Bender & Wu [14, 15, 16] studied the growth of coefficients for the quartic anharmonic oscillator in perturbation theory, and understood how instantons contribute
- Andre Voros [17]: non-perturbative structures in semiclassical expansions $\rightarrow$ development of the topic of exact WKB [18]
- Jean Écalle [19] worked out the mathematics of resurgent functions in the 80s. These concepts can be used to extract the non-perturbative information from the perturbative expansion and resum asymptotic series where Borel summability is obstructed.
- Gerald Dunne & Mithat Ünsal [20, 21] started to apply resurgence theory to Quantum Field Theories in the 2010s

Outline of the talks
Part I: Examples, asymptotic series, Borel summation, resurgent behaviour
Part II: Alien calculus, physical examples in QM

Literature:

Lecture notes:

- Daniele Dorigoni: An Introduction to Resurgence, Trans-Series and Alien Calculus [22]
- Marcos Mariño: An introduction to resurgence in quantum theory [23] (Basics of resurgence, Bender & Wu, Exact WKB, renormalons in QFT, basic numerical techniques with example Mathematica notebooks)
- Marco Serone: Lectures on Resurgence in Integrable Field Theories [24] (Overview, Stokes phenomenon in Airy function, Alien calculus, Integrable QFT)
  The Power of Perturbation Theory (Steepest descent analysis of quartic integrals, quantum mechanical examples, "exact perturbation theory") [25]
- Gerald Dunne: Introductory Lectures on Resurgence (Stokes phenomenon in Airy function, non-linear Stokes phenomenon, Heisenber-Euler QED, advanced numerical methods in asymptotic analysis) [26]
- Brent Pym: Resurgence in geometry and physics [27]

Review papers:

- I. Aniceto, G. Basar, R. Schiappa: A primer on resurgent transseries and their asymptotics [28]
- David Sauzin: Resurgent functions and splitting problems [29], Introduction to 1-summability and resurgence [30]

Theses:

- T. Reis: On the resurgence of renormalons in integrable theories [31]
- L. Schepers: Resurgence in deformed integrable models [32]

Books:

- R.B. Dingle: Asymptotic expansions: their derivation and interpretation [7]
- Jean Écalle: Les fonctions résurgentes (Vol. 1,2,3) [19], Guided tour through resurgence theory (short notes) [33]

Part I
Basics of asymptotics and examples for resurgence

1 Quartic integral

We study the following integral

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-S(x)}$$

with "action"

$$S(x) = \frac{1}{2} m x^2 + \frac{g}{4} x^4$$

as a 0-dimensional analogue for Euclidean path integrals of quartic interaction. The integral below

$$\int_{-\infty}^{\infty} \mathrm{d}x \; e^{-a x^{2p}} x^{2q} \quad \overset{t=ax^{2p}}{=} \quad \frac{1}{p a^{\frac{q+1/2}{p}}} \int_0^{\infty} \mathrm{d}t \, e^{-t} t^{\frac{q+1/2}{p}-1} = \frac{\Gamma\left(\frac{q+1/2}{p}\right)}{p a^{\frac{q+1/2}{p}}}$$

will be useful.

At first we assume $m, g > 0$ and we expand for small $g \ll 1$. This way we will arrive at

$$\begin{aligned} Z(g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-\frac{1}{2} m x^2} \sum_{n=0}^{\infty} \frac{(-g/4)^n}{n!} x^{4n} \sim \sum_{n=0}^{\infty} \frac{(-g/4)^n}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-\frac{1}{2} m x^2} x^{4n} \\ &\overset{p=1,q=2n}{=} \quad \frac{1}{\sqrt{m}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(2n + 1/2\right)}{\Gamma(1/2)\Gamma(n+1)} \left(\frac{g}{m^2}\right)^n \end{aligned}$$

where we exchanged the order of summation and integration (!)

For simplicity we may put $m = 1$. Then our expansion looks like

$$Z(g) \sim \sum_{n=0}^{\infty} c_n g^n = 1 - \frac{3}{4} g + \frac{105}{32} g^2 - \frac{3465}{128} g^3 + \frac{675675}{2048} g^4 + O(g^5)$$

with

$$c_n = \frac{(-1)^n \Gamma\left(2n + 1/2\right)}{\Gamma(1/2)\Gamma(n+1)}. \tag{1.1}$$

What is the radius of convergence? By the ratio test we have

$$\frac{\left|c_{n+1} g^{n+1}\right|}{\left|c_n g^n\right|} = \left|\frac{c_{n+1}}{c_n}\right| g < 1$$

that means

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)}{(2n+3/2)(2n+1/2)} = 0. \tag{1.2}$$

It is an asymptotic series, it cannot be summed up directly, we will study them in Section 2.

Let us examine the growth of the coefficients more closely. It is clear that they (1.1) grow factorially, with $\sim n!$. Using Stirling's approximation

$$\Gamma(n+\alpha) \sim \sqrt{2\pi} n^{n+\alpha-1/2} e^{-n}, \quad n \gg 1 \tag{1.3}$$

one can show that

$$c_n \sim \Gamma(n)\frac{(-4)^n}{\sqrt{2\pi}}$$

at leading order. What about subleading terms? Idea: organize subleading terms into subleading factorials:

$$c_n \sim \Gamma(n) + (\ldots)\Gamma(n-1) + (\ldots)\Gamma(n-2) + \ldots$$

or more precisely

$$c_n \sim \frac{1}{\sqrt{2\pi}}\left\{C_0\frac{\Gamma(n)}{(-1/4)^n} + C_1\frac{\Gamma(n-1)}{(-1/4)^{n-1}} + C_2\frac{\Gamma(n-2)}{(-1/4)^{n-2}} + C_3\frac{\Gamma(n-3)}{(-1/4)^{n-3}} + \ldots\right\}.$$

The corrections to the leading term then read as

$$c_n \sim \frac{\Gamma(n)}{\sqrt{2\pi}(-1/4)^n}\left\{C_0 + \frac{C_1(-1/4)}{(n-1)} + \frac{C_2(-1/4)^2}{(n-1)(n-2)} + \frac{C_3(-1/4)^3}{(n-1)(n-2)(n-3)} + \ldots\right\}.$$

To determine the C_k, one can use the Stirling series for the $\log\Gamma$ function

$$\ln\Gamma(n) = \frac{1}{2}\ln(2\pi) + \left(n-\frac{1}{2}\right)\ln n - n + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\frac{1}{n^{2k-1}}$$

where the B_n-s are the Bernoulli polynomials generated as

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!}.$$

Surprisingly

$$C_0 = 1 \qquad C_1 = \frac{3}{4} \qquad C_2 = \frac{105}{32} \qquad C_3 = \frac{3465}{128} \qquad C_4 = \frac{675675}{2048} \qquad \ldots$$

and we conjecture that

$$C_n = (-1)^n c_n \tag{1.4}$$

in general. The coefficients C_n that govern the subleading corrections to the large order behaviour of the original perturbative coefficients c_n are the same (up to alternating signs) as the low order c_n themselves. This is a manifestation of resurgence, but it is a special case, in general the relation is more complicated.

2 Asymptotic series

We have a formal power series

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n$$

that is only for bookkeeping the coefficients that are coming from an asymptotic expansion. (We cannot sum it up as it is not necessarily convergent.) It is asymptotic to a function

$$f(x) \sim \varphi(x), \quad \text{for} \quad x \to 0$$

if the difference of the finite truncation at N^{th} order to the function is of $O(x^{N+1})$ i.e.

$$\exists C_N : \quad \left| f(x) - \sum_{n=0}^{N} a_n x^n \right| \leq C_N x^{N+1}. \tag{2.1}$$

It is not unique (different functions may have the same asymptotic expansion), since the exponential term

$$e^{-1/x} = 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \dots$$

whose Taylor coefficients are zero (non-perturbative), cannot be captured by a power series:

$$e^{-1/x} \sim 0, \qquad\qquad \exists C_N : \quad e^{-1/x} \leq C_N x^N.$$

Typically in physics a_n-s grow factorially. The Gevrey order measures with which power of the factorial they grow, for

$$|a_n| \leq K c^n (n!)^m$$

we have Gevrey-$1/m$. In physics we usually have Gevrey-1, however, with a change of variables $x \to y^m$ and using Stirling formula (1.3) one can always transform the series to Gevrey-1 type $(n!)^m x^n \to (m \cdot n)! y^{mn}$.

As already noted in (1.2) their radius of convergence is zero. The terms shrink at first (exponential decay as $x^n = e^{-|\ln x| n}$) then the factorial wins. How to make sense of them?

- **Truncation after the first few terms**
  Based on (2.1) the error is polynomially small, but it only works for $x \ll 1$.

- **Optimal truncation**
  We keep going until the terms are getting smaller and smaller in magnitude, and stop at the smallest term. After this they diverge, spoiling our approximation. Where does this happen? Assuming

  $$a_n \sim n!$$

  we minimize the (logarithm of) the terms in n as a continuous variable:

  $$\frac{\mathrm{d}}{\mathrm{d}n} \cdot / \quad \ln(n! x^n) = n \ln n - n + n \ln x + O(\ln n)$$

  where we used Stirling formula again. This leads approximately to

  $$\ln n + \ln x = 0 \qquad\qquad \Rightarrow \qquad\qquad n_{\text{opt}} \sim \frac{1}{x}.$$

  Substituting this value back

  $$(n_{\text{opt}}!)\, x^{n_{\text{opt}}} \sim \left(1/x^{1/x} e^{-1/x}\right) x^{1/x} = e^{-1/x}$$

  shows that the error we make is exponentially small.
  Why is this justified? In general stopping at the smallest term is at best a rule of thumb, however for some cases it is proven to be indeed the optimal way to truncate [34].

- **Resummation**
  The idea of Borel transform is to divide with the factorial growth of the coefficients and define a new series

  $$\hat{\varphi}(t) \equiv \mathcal{B}[\varphi](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n, \tag{2.2}$$

  that is convergent: if $\varphi(x)$ is Gevrey-1, $\mathcal{B}[\varphi](t)$ has a finite radius of convergence. (For an already convergent series, the Borel transform leads to an entire function bounded by an exponential in every direction.) This means that we can perform the sum inside the radius.
  What to do with this? We can "restore" the factorial growth by the Γ-function integral

  $$\int_0^\infty \mathrm{d}t\, e^{-t/x} t^n = n! x^{n+1}$$

for every term in (2.2), i.e. by a Laplace transform applied on the Borel transform:

$$\mathcal{L}[\mathcal{B}[\varphi]](x) = x^{-1} \int_0^\infty \mathrm{d}t \, e^{-t/x} \mathcal{B}[\varphi](t) = \int_0^\infty \mathrm{d}t \, e^{-t} \mathcal{B}[\varphi](tx). \tag{2.3}$$

For this one has to analytically continue the Borel transform outside of its convergence radius, to be able to integrate along the positive real line to infinity. This sequence of transformations

$$\mathcal{S} = \mathcal{L} \circ \mathcal{B}$$

is the Borel summation.

Remark: We could have also defined the summation as

$$\mathcal{B}[\varphi](t) = \sum_{n=0}^\infty \frac{a_{n+1}}{n!} t^n, \qquad \mathcal{S}[\varphi](x) = a_0 + \int_0^\infty \mathrm{d}t \, e^{-t/x} \mathcal{B}[\varphi](t)$$

by separating the constant term. This is also a good definition if we have finitely many negative powers of x as well (finite additivity property of summations).

3 Singularities of the Borel plane

Let us then return to the asymptotic expansion of the quartic integral and apply the Borel transform to it. It has a closed form, but without any knowledge of special functions we can analyze the asymptotics of the coefficients. The leading order

$$\mathcal{B}[Z](t) \sim \sum_{n=1}^\infty \frac{\Gamma(n)}{\Gamma(n+1)} \frac{(-4)^n}{\sqrt{2\pi}} t^n = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^\infty \frac{(-4t)^n}{n} = -\frac{\ln(1+4t)}{\sqrt{2\pi}}$$

leads to a singularity at $t_0 = -1/4$ on the complex plane of the variable t, the so-called Borel plane. What about the sub-leading terms? For C_k we have

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} / \cdot \qquad \sum_{n=k+1}^\infty \frac{\Gamma(n-k)}{\Gamma(n+1)} \frac{t^n}{(-1/4)^{n-k}} \sim \sum_{n=k+1}^\infty \frac{1}{n(n-1)\dots(n-k)} \frac{t^n}{t_0^{n-1}}$$

and if we differentiate this sum k times we arrive at

$$\sum_{n=k+1}^\infty \frac{1}{(n-k)} \frac{t^{n-k}}{t_0^{n-k}} = -\ln(1-t/t_0) = -\ln(t_0-t) + \text{const.}$$

We calculate the inverse operation of the derviation, the indefinite integral

$$-\int \mathrm{d}t \, 1 \ln(t_0-t) = \int \mathrm{d}t \, \frac{\mathrm{d}(t_0-t)}{\mathrm{d}t} \ln(t_0-t) = (t_0-t)\ln(t_0-t) + \overbrace{\int \mathrm{d}t \, (t_0-t) \frac{\mathrm{d}}{\mathrm{d}t} \ln(t_0-t)}^{t}$$

and we get after integrating by parts once. We focus only on the singular part, and integrate it by parts once more:

$$\int \mathrm{d}t \, (t_0-t)\ln(t_0-t) = -\int \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}(t_0-t)^2\right] \ln(t_0-t) = -\frac{1}{2}(t_0-t)^2 \ln(t_0-t) + \text{regular}.$$

After k integration we arrive at

$$\overbrace{\int \mathrm{d}t \, \dots \int}^{k} \mathrm{d}t \, [-\ln(t_0-t)] = -\frac{1}{k!}(t-t_0)^k \ln(t_0-t) + \text{regular}.$$

for the singular part. In summary this means

$$\mathcal{B}[Z](t) \sim -\left[\sum_{k=0}^\infty \frac{C_k}{k!}(t-t_0)^k\right] \ln(t_0-t) + \text{regular}.$$

Using the relation between the asymptotic coefficients C_k and the perturbative coefficients c_k in (1.4) we have

$$\hat{Z}(t) \sim -\hat{Z}(t_0-t)\ln(t_0-t) + \text{regular},$$

that is, the logarithmic singularity of the Borel plane is multiplied by a function, whose expansion around t_0 happens to coincide with the original expansion around zero, evaluated at negative arguments $\hat{Z}(-t)$.

4 Uniqueness of Borel resummation

Let us now illustrate the idea of Borel resummation on an example by Euler [35]. We will also use the variable

$$z = 1/x, \qquad \varphi(z) = \sum_n a_n z^{-n}$$

and treat our asymptotic expansion that is valid for large $z \gg 1$ instead of small $x \ll 1$.
We consider the Euler ODE

$$-f'(z) + f(z) = \frac{1}{z} \tag{4.1}$$

and substitute the ansatz

$$f(z) \sim \varphi(z) = \sum_{n=0}^{\infty} a_n z^{-n-1},$$

that is

$$\sum_{n=1}^{\infty} n a_{n-1} z^{-n-1} + \sum_{n=0}^{\infty} a_n z^{-n-1} = z^{-1},$$

after shifting the first sum. Now this leads to the simple recursion

$$a_n = -n a_{n-1}, \quad a_0 = 1$$

for the coefficients that is solved exactly by

$$a_n = (-1)^n n!$$

(Euler exactly studied this type of hypergeometric series

$$0! - 1! + 2! - 3! + 4! - 5! + \dots,$$

where the ratio of consecutive terms is not constant, but is a rational function of n, in this case

$$\frac{a_n}{a_{n-1}} = -n.)$$

Its Borel transform is then a simple geometric series

$$\mathcal{B}[\varphi](t) = \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t}$$

that is convergent for $|t| < 1$ due to the simple pole singularity at $t = -1$. It can be continued analytically for the whole positive real line $t \geq 0$, and the Laplace integral (2.3) is well defined and reads

$$\mathcal{S}[\varphi](z) = \int_0^{\infty} \mathrm{d}t \, \frac{e^{-tz}}{1+t} \overset{s=(1+t)z}{=} e^z \int_z^{\infty} \mathrm{d}s \, \frac{e^{-s}}{s} = e^z \Gamma(0, z) \tag{4.2}$$

where

$$\Gamma(n, z) = \int_z^{\infty} \mathrm{d}s \, s^{n-1} e^{-s}$$

is the incomplete gamma function.

If we re-expand

$$\mathcal{S}[\varphi](z) \overset{\tau=tz}{=} \int_0^{\infty} \mathrm{d}\tau \, \frac{e^{-\tau}}{z+\tau} \sim \frac{1}{z} \sum_{n=0}^{\infty} \int_0^{\infty} \mathrm{d}\tau \, e^{-\tau} (-\tau/z)^n = \sum_{n=0}^{\infty} (-1)^n n! z^{-n-1}$$

we get the same asymptotic series, but is this the correct solution to the ODE in (4.1)? That is

$$f(z) \overset{?}{=} \mathcal{S}[\varphi](z). \tag{4.3}$$

It is certainly a particular solution

$$-\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{S}[\varphi](z)+\mathcal{S}[\varphi](z)=\int_0^\infty \mathrm{d}t\,\frac{(t+1)e^{-tz}}{1+t}=\int_0^\infty \mathrm{d}t\,e^{-tz}=\frac{1}{z}\quad\checkmark$$

but we could add the homogeneous solution to it

$$f_h'(z)=f_h(z)\qquad\Rightarrow\qquad f_h(z)=ce^z.$$

What are the initial/boundary conditions? We have

$$\lim_{z\to\infty}\mathcal{S}[\varphi](z)=0,\qquad \lim_{z\to\infty}e^z=\infty$$

Thus if we require that $f(z)$ vanishes for $z\to\infty$ we have indeed equality in (4.3).

Discontinuity of the incomplete Γ

The idea is to continue the integral (4.2) analytically in z by rotating its value on the complex plane from positive to negative by changing its complex argument by π. For this we have to counter-rotate the integration contour in t, such that the product in the exponent e^{-tz} remains positive $tz>0$, that is

$$\mathcal{S}[\varphi](z)=\int_0^{e^{-i\theta}\infty}\mathrm{d}t\,\frac{e^{-tz}}{1+t},\qquad \theta=\arg z.$$

Or rather, for a rotation from positive z as $z\to e^{i\theta}z$ we have

$$\mathcal{S}[\varphi](e^{i\theta}z)=\int_0^{e^{-i\theta}\infty}\mathrm{d}t\,\frac{e^{-te^{i\theta}z}}{1+t},\qquad z>0.$$

We can then reach the negative line in two ways, by rotating the argument of z by $\theta=\pm\pi$ and thus counter-rotating in t by $\mp\pi$. The difference between the two

$$\mathcal{S}[\varphi](e^{i\pi}z)-\mathcal{S}[\varphi](e^{-i\pi}z)=\int_0^{e^{-i\pi}\infty}\mathrm{d}t\,\frac{e^{tz}}{1+t}-\int_0^{e^{i\pi}\infty}\mathrm{d}t\,\frac{e^{tz}}{1+t}=-2\pi i\mathrm{Res}_{t=-1}\frac{e^{tz}}{1+t}=-2\pi ie^{-z},\qquad z>0$$

is exactly the residue picked up when combining the two contours encircling the pole in the Borel transform. This is a non-perturbative connection formula for the incomplete gamma function

$$e^{-z}\Gamma(0,e^{i\pi}z)-e^{-z}\Gamma(0,e^{-i\pi}z)=-2\pi ie^{-z},$$

which has a branch cut along the negative real line $z<0$, and this is exactly its discontinuity along this cut.

5 Ambiguities

Let us change the sign of the derivative term in the Euler ODE (4.1), i.e.

$$+f'(z)+f(z)=\frac{1}{z}. \tag{5.1}$$

This changes the coefficients from alternating to non-alternating

$$a_n=n!,$$

and so the Borel transform

$$\mathcal{B}[\varphi](t)=\sum_{n=0}^\infty t^n=\frac{1}{1-t}$$

has a pole at $t=1$, i.e. we have a singularity along the integration line (!) in the summation formula

$$\mathcal{S}[\varphi](z)=\int_0^\infty \mathrm{d}t\,\frac{e^{-tz}}{1-t}=? \tag{5.2}$$

The simple idea is to define the integral in the principal value sense

$$\mathcal{S}[\varphi](z) = \text{P.V.}\int_0^\infty \mathrm{d}t\,\frac{e^{-tz}}{1-t} \overset{s=(t-1)z}{=} -e^{-z}\text{P.V.}\int_{-z}^\infty \mathrm{d}s\,\frac{e^{-s}}{s} = e^{-z}\text{Ei}(z), \tag{5.3}$$

where

$$\text{Ei}(z) = -\text{P.V.}\int_{-z}^\infty \mathrm{d}s\,\frac{e^{-s}}{s}$$

is the exponential integral function. We also introduce the directional Laplace transform

$$\mathcal{L}_\theta[\hat{\varphi}](z) = \int_0^{e^{i\theta}\infty} \mathrm{d}t\, e^{-tz}\hat{\varphi}(t)$$

and the so-called lateral Borel resummations

$$\mathcal{S}_\theta^\pm[\varphi](z) = \mathcal{L}_{\theta\pm 0}[\mathcal{B}[\varphi]](z) = \int_0^{e^{i(\theta\pm 0)}\infty} \mathrm{d}t\, e^{-tz}\mathcal{B}[\varphi](t),$$

where $\mathcal{S}^\pm \equiv \mathcal{S}_0^\pm$ for $\theta = 0$, i.e. when the integral is originally along the real line. In our case, the difference between the two is

$$\left(\mathcal{S}^+ - \mathcal{S}^-\right)[\varphi](z) = -2\pi i \text{Res}_{t=1}\frac{e^{-tz}}{1-t} = 2\pi i e^{-z},$$

or compared to the principal value we have

$$\mathcal{S}^\pm[\varphi] = \mathcal{S}[\varphi] \pm i\pi e^{-z}. \tag{5.4}$$

Once again, (5.3) is a particular solution to the ode, but the general solution should include the solution to the homogeneous equation

$$f_h'(z) = -f_h(z) \qquad \Rightarrow \qquad f_h(z) = ce^{-z}$$

such that

$$f(z) = \mathcal{S}[\varphi](z) + ce^{-z}, \quad c \in \mathbb{R}, \tag{5.5}$$

for some constant to satisfy initial conditions at some $0 < z_0 < \infty$ (in this case the requirement that $\lim_{z\to\infty} f(z) = 0$ does not fix c).

Notice that the homogenous solution has the same form as the imaginary ambiguity in (5.4). It is an exponentially small correction that is not captured by the perturbative series, however it is important to recover the set of all solutions to the ODE, so it seems this term has to be included in the asymptotic expansion in some way. There is a formal way to this, by defining a trans-series that encodes both the perturbative part and the non-perturbative part:

$$f(z) \sim \sum_{n=0}^\infty n! z^{-n-1} + \tilde{c}e^{-z}.$$

To arrive at the general solution (5.5) we have to choose $\tilde{c}$ such that depending on the lateral resummation (5.4) we arrive at the same value

$$f(z) = \left(\mathcal{S}^\pm[\varphi] \mp i\pi e^{-z}\right) + ce^{-z} = \mathcal{S}^\pm\Big[\sum_{n=0}^\infty n! z^{-n-1}\Big] + \overbrace{(c \mp i\pi)}^{\tilde{c}}\, e^{-z}.$$

In the end we have the trans-series

$$f(z) \sim \tilde{\varphi}(z) = \sum_{n=0}^\infty n! z^{-n-1} + (c \mp i\pi)\, e^{-z}$$

that has to be resummed

$$f(z) = \mathcal{S}^\pm[\tilde{\varphi}](z)$$

where $\mathcal{S}^\pm[e^{-z}] = e^{-z}$. Importantly, the real part c of the exponentially small term is fixed by external information (the initial conditions), while the imaginary part $\mp i\pi$ is related to the asymptotic growth of the coefficients and

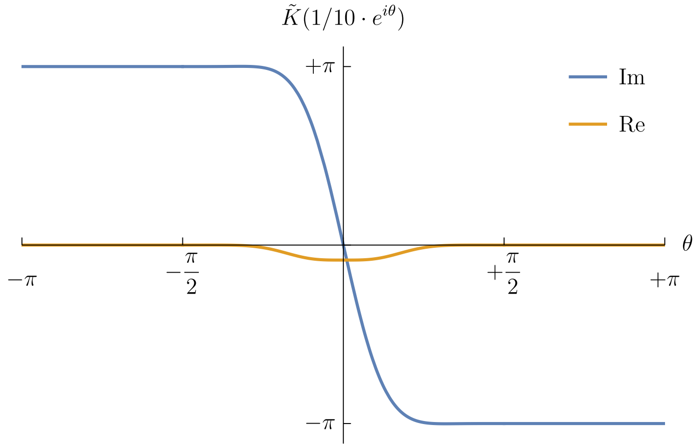

Figure 5.1: Behaviour of $K(re^{i\theta})$ as a function of θ.

the singularities of the Borel plane. The appearance of the ambiguities is the signal of the Stokes-phenomenon (see later).

Let us stick to the case $c = 0$. It is interesting to note that the difference between the optimal truncation and the exact result can be quantitatively studied. In this case we may define the "amplitude" of the exponentially small error as $K(z)$

$$f(z) = e^{-z}\mathrm{Ei}(z) = \sum_{n=0}^{n_{\mathrm{opt}}} n! z^{-n-1} + K(z)e^{-z}.$$

If we plot $K(re^{i\theta})$ for some large value of r as a function of θ, its imaginary part smoothly interpolates between $+i\pi$ and $-i\pi$ and hits 0 for $\theta = 0$ (see Figure 5.1). This is related to Stokes phenomenon where exponentially small contributions are switching on and off in a smooth way, see also "error function behaviour" in [34].

6 Stokes phenomenon

6.1 Airy function

We consider a quantum mechanical particle in a linear (gravitational) potential:

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}z^2} + V(z), \qquad\qquad V(z) = \begin{cases} mgz & z > 0 \\ \infty & z \leq 0 \end{cases}$$

The time independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\psi''(z) + mgz\psi(z) = E\psi(z).$$

Reordering the terms

$$\psi''(z) = \underbrace{\frac{2m^2 g}{\hbar^2}}_{\alpha^3}\Bigg(z - \underbrace{\frac{E}{mg}}_{z_0}\Bigg)\psi(z),$$

and introducing the change of variables

$$x \equiv \alpha(z - z_0)$$

where $z_0 = \frac{E}{mg}$, $\alpha = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}$, leads to the Airy equation

$$\boxed{\psi''(x) = x\psi(x)}. \tag{6.1}$$

In Fourier space

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \tilde{\psi}(k) e^{ikx}$$

the same equation reads

$$ik^2\tilde{\psi}(k) = \tilde{\psi}'(k) \qquad \Rightarrow \qquad \tilde{\psi}(k) = e^{i\frac{k^3}{3}}$$

thus we arrive at the integral representation of the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{i\left(\frac{k^3}{3}+kx\right)}.$$

6.2 WKB - Asymptotic expansion

We consider the time independent Schrödinger equation for a one-dimensional potential problem

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

and treat $\hbar$ as a small parameter. The second order ODE can be interpreted as a singular perturbation problem:

$$\hbar^2\psi''(x) + p^2(x)\psi(x) = 0$$

where we define the classical momentum as

$$p(x) \equiv \sqrt{2m(E-V(x)}.$$

We extend the eikonal approximation

$$\psi(x) = e^{\frac{i}{\hbar}S(x)}$$

such that we write the exponent as a formal power series in $\hbar$:

$$\psi(x) = \exp\left(\frac{i}{\hbar}\int^x Y(x')\mathrm{d}x'\right), \quad S'(x) = Y(x) = \sum_{n=0}^{\infty} Y_n(x)\hbar^n.$$

This leads to a Ricatti-type equation for $Y(x)$

$$i\hbar Y'(x) - Y^2(x) + p^2(x) = 0,$$

and expanding to leading and next-to-leading orders gives

$$-Y_0^2(x) + p^2(x) = 0 \qquad \Rightarrow \qquad Y_0(x) = \pm p(x)$$

$$iY_0'(x) - 2Y_0(x)Y_1(x) = 0 \qquad \Rightarrow \qquad Y_1(x) = \frac{iY_0'(x)}{2Y_0(x)} = i\frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{p(x)}$$

which leads to the two independent solutions

$$\psi_\pm(x) = \frac{1}{\sqrt{p(x)}} e^{\pm\frac{i}{\hbar}\int^x p(x')\mathrm{d}x}\left(1 + O(\hbar)\right).$$

Let us now turn back to the Airy equation (6.1) where

$$p^2(x) = -x$$

and the asymptotic behaviour of the solutions is

$$\psi_\pm(x) = \frac{1}{x^{1/4}} e^{\pm\frac{1}{\hbar}\frac{2}{3}x^{3/2}}\left(1 + O(\hbar)\right).$$

Instead proceeding by solving for the system of $Y_n(x)$-s systematically, we simply substitute this asymptotics back into the ODE (6.1). Using the variable

$$z = \frac{4}{3}x^{3/2}$$

we write the solution as

$$\psi_\pm(x) = \left(\frac{3}{4}z(x)\right)^{-1/6} e^{\pm\frac{z(x)}{2\hbar}} f_\pm(z(x))$$

together with the functions $f_\pm(z)$ encoding the corrections. These satisfy the ODE

$$\hbar^2 f''_\pm(z) \pm \hbar f'_\pm(z) + \frac{5\hbar^2}{36z^2} f_\pm(z) = 0.$$

After rescaling $z \to z/\hbar$ (no explicit dependence on $\hbar$ anymore) we can conclude that small $\hbar$ expansion goes hand-in-hand with large z expansion. Taking the ansatz

$$f_\pm(z) \sim \varphi_\pm(z) = \sum_{n=0}^{\infty} a_n^\pm z^{-n}$$

leads to the recursion

$$(n+1)a_{n+1}^\pm \mp \left(n(n+1) + \frac{5}{36}\right) a_n^\pm = 0$$

for the coefficients. This is solved by the product

$$a_{n+1}^\pm = (\pm 1)^{n+1} \prod_{k=0}^{n} \left(\frac{k(k+1) + \frac{5}{36}}{k+1}\right) a_0^\pm.$$

Note that a combination of gamma functions

$$\frac{\Gamma((k+1)+\alpha)\Gamma((k+1)+\beta)}{\Gamma((k+1)+1)} = \frac{(k+\alpha)(k+\beta)}{(k+1)} \times \frac{\Gamma(k+\alpha)\Gamma(k+\beta)}{\Gamma(k+1)}$$

will behave exactly this way, if we have

$$(k+\alpha)(k+\beta) \overset{!}{=} k(k+1) + \frac{5}{36} \qquad \Rightarrow \qquad \alpha + \beta = 1, \qquad \alpha\beta = \frac{5}{36}$$

that is $\alpha = 1 - \beta$ and $\beta = \frac{1}{6}$. Thus we arrive at the explicit form of the coefficients

$$a_n^\pm = (\pm 1)^n \frac{\Gamma(n+1-\beta)\Gamma(n+\beta)}{\Gamma(1-\beta)\Gamma(\beta)\Gamma(n+1)} = (\pm 1)^n \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{2\pi\Gamma(n+1)}$$

if we also normalize them to unity $a_0^\pm = 1$ at $n = 0$, where the reflection formula gives simply

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)} \qquad \Rightarrow \qquad \Gamma(1-\beta)\Gamma(\beta)|_{\beta=1/6} = 2\pi.$$

Let us now analyze the large n behaviour of the coefficients. Using Stirling's approximation

$$\Gamma(n+\alpha) \sim \sqrt{2\pi} n^{n+\alpha-1/2} e^{-n}$$

the leading behaviour is

$$a_n^\pm \sim (\pm 1)^n \frac{\Gamma(n)}{2\pi}$$

which means they are growing factorially. Asymptotic series again! The subleading corrections to the coefficients are

$$a_n^\pm = (\pm 1)^n \frac{\Gamma(n)}{2\pi} \left\{1 \mp \frac{5}{72}\frac{(\pm 1)^1}{(n-1)} + \frac{385}{10368}\frac{(\pm 1)^2}{(n-1)(n-2)} + \dots\right\}$$

and since the two series are related as

$$a_n^+ = (-1)^n a_n^-$$

and start as

$$a_n^{\pm} = \left\{1, \pm\frac{5}{72}, \pm\frac{385}{10368}, \dots\right\},$$

we have

$$a_n^{\pm} = \frac{1}{2\pi}\left\{a_0^{\mp}\frac{\Gamma(n)}{(\pm 1)^n} + a_1^{\mp}\frac{\Gamma(n-1)}{(\pm 1)^{n-1}} + a_2^{\mp}\frac{\Gamma(n-2)}{(\pm 1)^{n-2}} + a_3^{\mp}\frac{\Gamma(n-3)}{(\pm 1)^{n-3}} + \dots\right\}.$$

It can be interpreted that the large order behaviour of a_n^+ is influenced by the low order coefficients of a_n^- and vice versa.

6.3 Connection formulae

We investigate the Borel transform that can be performed explicitely

$$\hat{\varphi}_{\pm}(t) = \mathcal{B}[\varphi_{\pm}](t) = \sum_{n=0}^{\infty}\frac{a_n^{\pm}t^n}{n!} = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \pm t\right)$$

in terms of the hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1).$$

Note that

$$\hat{\varphi}_-(t) = \hat{\varphi}_+(-t).$$

The leading order growth of the coefficients gives

$$\sum_{n=0}^{\infty}(\pm 1)^n\frac{t^n}{2\pi n} \sim -\frac{\ln(1\mp t)}{2\pi}$$

that is, logarithmic singularities appear at $t_0 = \pm 1$. The large order/low order relations mean that

$$\hat{\varphi}_{\pm}(t) \sim -\frac{1}{2\pi}\hat{\varphi}_{\mp}(t\mp 1)\log(1\mp t) + \text{regular}.$$

If we use the discontinuity of the logarithm

$$\log(1-(t\pm i0)) = \ln|1-t| \mp i\pi, \quad t \geq 1$$

we have

$$\hat{\varphi}_+(t+i0) - \hat{\varphi}_+(t-i0) = i\hat{\varphi}_-(t-1) = i\hat{\varphi}_+(1-t). \tag{6.2}$$

(This translates directly into the connection formula for hypergeometric functions

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; t+i0\right) - {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; t-i0\right) = i\,{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1-t\right), \quad t \geq 1.)$$

The Airy function is the inverse Borel transform of the $\hat{\varphi}_-(t)$, that is not singular along the positive real line:

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{z(x)}{2}}\mathcal{S}[\hat{\varphi}_-](z), \quad x > 0 \tag{6.3}$$

If we rotate the argument of x by $\theta = \pm\frac{2\pi}{3}$, it translates to rotating z by $\pm\pi$, and so counter-rotating t by $\mp\pi$. That is, after rotating the inverse Borel transforms according to (6.2) we get

$$e^z\mathcal{S}[\hat{\varphi}_-](e^{i\pi}z) - e^z\mathcal{S}[\hat{\varphi}_-](e^{-i\pi}z) = i\mathcal{S}[\hat{\varphi}_-](z),$$

and finally, for the Airy function this means the non-perturbative connection formula

$$\mathrm{Ai}(e^{\frac{2\pi}{3}i}x)e^{i\pi/6} - \mathrm{Ai}(e^{-\frac{2\pi}{3}i}x)e^{-i\pi/6} = i\mathrm{Ai}(x). \tag{6.4}$$

Similarly as before, we may define the resummation of $\mathcal{S}[\hat{\varphi}_+](z)$ in the principal value sense, by taking the sum of the two lateral resummations:

$$\mathcal{S}[\hat{\varphi}_+](z) \equiv \int_0^\infty \mathrm{d}t\; e^{-zt}\,\frac{\hat{\varphi}_+(t+i0)+\hat{\varphi}_+(t-i0)}{2} = \frac{1}{2}\left(\mathcal{S}[\hat{\varphi}_-](e^{i\pi}z)+\mathcal{S}[\hat{\varphi}_-](e^{-i\pi}z)\right). \tag{6.5}$$

With the same prefactors as in (6.3) this turns out to be the other well known solution to the differential equation

$$\frac{1}{2}\mathrm{Bi}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}e^{z/2}\mathcal{S}[\hat{\varphi}_+](z), \quad x>0$$

and (6.5) translates to the connection formula for $\mathrm{Bi}(x)$:

$$\mathrm{Ai}(e^{2\pi/3i}x)e^{i\pi/6} + \mathrm{Ai}(e^{-2\pi/3i}x)e^{-i\pi/6} = \mathrm{Bi}(x). \tag{6.6}$$

If we combine (6.4) and (6.6) we get

$$\mathrm{Ai}(e^{\pm\frac{2\pi}{3}i}x) = \frac{e^{\mp i\pi/6}\mathrm{Bi}(x) + ie^{\pm i\pi/3}\mathrm{Ai}(x)}{2}. \tag{6.7}$$

Perturbatively, the expansions of the $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}\left\{\sum_{n=0}^\infty(-1)^n a_n^+\left(\frac{4}{3}x^{2/3}\right)^{-n}\right\}$$
$$\frac{1}{2}\mathrm{Bi}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}e^{\frac{2}{3}x^{3/2}}\left\{\sum_{n=0}^\infty a_n^+\left(\frac{4}{3}x^{2/3}\right)^{-n}\right\}$$

and we conclude that the perturbative part on the Stokes line should be

$$\mathrm{Ai}(e^{\pm\frac{2\pi}{3}i}x) \sim \frac{e^{\mp i\pi/6}\mathrm{Bi}(x)}{2}.$$

However (6.7) shows that we need also a non-perturbative correction term, that is suppressed by $e^{-\frac{4}{3}x^{3/2}}$ compared to the dominant contribution. This shows that to represent $\mathrm{Ai}(e^{\pm\frac{2\pi}{3}i}x)$ (on the Stokes line) we need a trans-series.

6.4 Steepest descent contours

We would like to understand the previous connection formulae from the integral definition of the Airy function. We evaluate it for large absolute values of x in any direction $x = re^{i\theta}$, i.e. $r\to\infty$. We perform a change of variables

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\, e^{i\left(\frac{k^3}{3}+kx\right)} \overset{k=-iq\sqrt{r}}{=} \frac{\sqrt{r}}{2\pi i}\int_{-i\infty}^{i\infty}\mathrm{d}q\, e^{r^{3/2}\left(-\frac{q^3}{3}+qe^{i\theta}\right)}$$

where based on the exponent we define the "action"

$$S(q) = \frac{q^3}{3} - qe^{i\theta},$$

such that

$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i}\int_{-i\infty}^{i\infty}\mathrm{d}q\, e^{-r^{3/2}S(q)}.$$

We could do saddle point approximations, etc. and it would lead to the same expansions we obtained from WKB. But how to evaluate this in a non-perturbative way? We may deform the contour in any way we want, since the exponent is an entire function. But the integral itself is well defined only, if

$$q^3 > 0$$

that means, one can only integrate along such contours that tend to infinity only at angles

$$\arg q = 0, \pm\frac{2\pi}{3}.$$

The idea of steepest descent is to keep the imaginary part of $S(q)$ fixed, and define contours such that along these the real part of $S(q)$ decreases monotonically. We define a flow time τ such that it parametrizes the curve $q(\tau)$. For this we choose

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = -\overline{S'(q)} = -\bar{S}'(\bar{q})$$

since then we have

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Im}S(q(\tau)) = S'(q)\frac{\mathrm{d}q}{\mathrm{d}\tau} - \bar{S}'(\bar{q})\frac{\mathrm{d}\bar{q}}{\mathrm{d}\tau} = 0$$

and also

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Re}S(q(\tau)) = S'(q)\frac{\mathrm{d}q}{\mathrm{d}\tau} + \bar{S}'(\bar{q})\frac{\mathrm{d}\bar{q}}{\mathrm{d}\tau} = -2\left|S'(q)\right|^2 < 0.$$

The fixed points of this flow are exactly at the stationary points of the integral

$$S'(q^*) = 0.$$

In our case

$$S'(q) = q^2 - e^{i\theta}$$

thus the flow reads

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = -\left(\bar{q}^2 - e^{-i\theta}\right)$$

and the fixed points are at

$$q^* = \pm e^{i\theta/2} \quad \Rightarrow \quad S(q^*) = \mp\frac{2}{3}e^{i3\theta/2}.$$

How does this flow look like? We introduce

$$q = u + iv$$

then the flow equation reads

$$\begin{aligned}\dot{u} &= -u^2 + v^2 + \cos\theta \\ \dot{v} &= 2uv - \sin\theta.\end{aligned}$$

The linearized equations around the fixed points are

$$\delta\dot{q} = \mp 2e^{-i\theta/2}\delta\bar{q},$$

or in coordinates

$$\begin{pmatrix}\delta\dot{u}\\ \delta\dot{v}\end{pmatrix} = \mp 2\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ -\sin\left(\frac{\theta}{2}\right) & -\cos\left(\frac{\theta}{2}\right)\end{pmatrix}\begin{pmatrix}\delta u\\ \delta v\end{pmatrix} \equiv 2M\begin{pmatrix}\delta u\\ \delta v\end{pmatrix},$$

where we defined the matrix M.

For $\theta = 0$ we have

$$q^* = \pm 1 \quad \Rightarrow \quad S(q^*) = \mp\frac{2}{3}$$

and the eigenvectors for the ± 1 eigenvalues are

$$M = \begin{pmatrix}\mp 1 & 0\\ 0 & \pm 1\end{pmatrix}, \quad \pm 1: \quad \begin{pmatrix}0\\ 1\end{pmatrix}, \quad \mp 1: \quad \begin{pmatrix}1\\ 0\end{pmatrix}.$$

For $\theta = \pi$

$$q^* = \pm i$$

$$M = \begin{pmatrix}0 & \pm 1\\ \pm 1 & 0\end{pmatrix}, \quad \pm 1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, \quad \mp 1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

and for $\theta = \frac{2\pi}{3}$

$$q^* = \pm e^{i\pi/3}$$

$$M = \begin{pmatrix}\mp\frac{1}{2} & \pm\frac{\sqrt{3}}{2}\\ \pm\frac{\sqrt{3}}{2} & \pm\frac{1}{2}\end{pmatrix}, \quad \pm 1: \quad \frac{1}{2}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}, \quad \mp 1: \quad \frac{1}{2}\begin{pmatrix}-\sqrt{3}\\ 1\end{pmatrix}$$

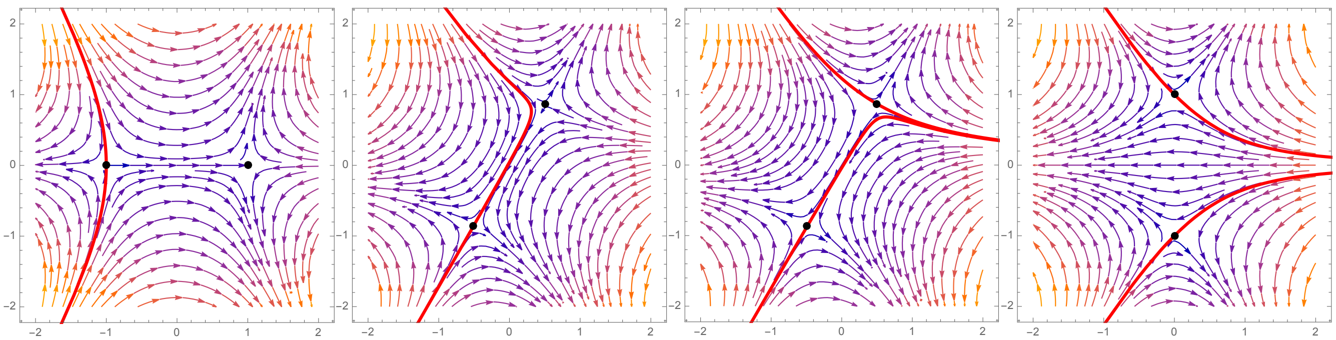

Figure 6.1: Flow for $\theta = 0,\ 2\pi/3 - \epsilon,\ 2\pi/3 + \epsilon,\ \pi$ respectively. [26]

as eigenvectors. After drawing the flow, one can draw the integration contours, by fixing the imaginary part of the action to the value at the saddle. Figure 6.1 shows that the integration contour jumps at $\theta = 2\pi/3$ and splits into two, exactly when it hits another saddle point, and where the Borel integral hits singularities according to Subsection 6.3.

Around the saddle points we may approximate the integral with a Gaussian. Up to quadratic order the exponent looks

$$S(q) = S(q^*) + q^*\delta q^2 + O(\delta q^3).$$

To define the integral properly we need the sign of the quadratic term to be positive

$$q^*\delta q^2 > 0$$

therefore we introduce

$$\delta q = -\left(q^*\right)^{-1/2}\xi, \quad \xi \in \mathbb{R}.$$

Performing the Gaussian gives

$$\begin{aligned}\mathrm{Ai}(x) &= e^{-r^{3/2}S(q^*)}\frac{\sqrt{r}}{2\pi i}\int_C \mathrm{d}\delta q\, e^{-r^{3/2}\xi^2} = e^{-r^{3/2}S(q^*)}\frac{\sqrt{r}}{2\pi i}\left(-\left(q^*\right)^{1/2}\right)\int_{-\infty}^{\infty}\mathrm{d}\xi\, e^{-r^{3/2}\xi^2} \\ &= \frac{1}{2\sqrt{\pi}}e^{\pm\frac{2}{3}x^{3/2}}\frac{1}{ir^{1/4}q^{*1/2}} = \begin{cases}\frac{1}{2\sqrt{\pi}x^{1/4}}e^{\frac{2}{3}x^{3/2}}i & q^* = +e^{i\theta/2} \\ \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}} & q^* = -e^{i\theta/2}\end{cases}.\end{aligned}$$

For $\theta = 0$, i.e. $x > 0$ we have only the $q^* = -1$ saddle contributing, since this has $S(-1) = 2/3 > 0$, and thus

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}} + \text{corr.}$$

For $\theta = \pi$, i.e. $x < 0$ both saddles contribute, and so

$$\mathrm{Ai}(-|x|) = \frac{1}{2\sqrt{\pi}|x|^{1/4}}\left(e^{i\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}e^{-i\frac{2}{3}|x|^{3/2}} + e^{-i\frac{\pi}{4}}e^{i\frac{2}{3}|x|^{3/2}}\right) = \frac{\cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)}{\sqrt{\pi}|x|^{1/4}} + \text{corr.}$$

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