

GRAVITATION AND VACUUM ENTANGLEMENT ENTROPY*

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The vacuum of quantum fields contains correlated fluctuations. When restricted to one side of a surface these have a huge entropy of entanglement that scales with the surface area. If UV physics renders this entropy finite, then a thermodynamic argument implies the existence of gravity. That is, the causal structure of spacetime must be dynamical and governed by the Einstein equation with Newton's constant inversely proportional to the entropy density. Conversely, the existence of gravity makes the entanglement entropy finite. This thermodynamic reasoning is powerful despite the lack of a detailed description of the dynamics at the cutoff scale, but it has its limitations. In particular, we should not expect to understand corrections to Einstein gravity in this way.

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In the vacuum state of a quantum field, fluctuations are correlated at spacelike separations. This entanglement implies that, although the vacuum is a pure quantum state, its restriction to a localized region is mixed. The corresponding entropy is dominated by the shortest wavelength modes, and scales as the area of the region boundary.^{1–4} In an ultraviolet (UV) complete relativistic quantum field theory on a *fixed* spacetime the entanglement entropy is infinite.

More specifically, in Minkowski spacetime, the vacuum of a relativistic quantum field restricted to the wedge $z > |t|$, lying to one side of an infinite xy plane, is a thermal state at temperature $T = \hbar/2\pi$ with respect to the Hamiltonian that generates Lorentz boosts (hyperbolic rotations) normal to the plane.^{5–7, a} (An observer

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^aI adopt units with the speed of light equal to one, and assume for concreteness that the spacetime is four-dimensional.

localized on a particular worldline with uniform acceleration a has proper time equal to the hyperbolic angle divided by a , so the state is thermal at the Unruh temperature $T_U = \hbar a/2\pi$ with respect to the Hamiltonian generating his proper time translations.) The entropy is infinite on account of the arbitrarily short wavelength fluctuations close to the horizon $z = t = 0$, which are entangled with partners similarly close on the other side of the horizon. If the contributions are cut off at a length ℓ_c , one obtains an entropy that scales as A/ℓ_c^2 , the area of the plane in units of the cutoff length. There is nothing inherently sick in the notion of infinite horizon entropy; on the contrary, what is puzzling is how horizon entropy could ever be *finite*.

Remarkably, however, the assumption that horizon entanglement entropy is somehow rendered finite by UV physics implies that the spacetime causal structure is dynamical, and that the metric satisfies Einstein's equation as a thermodynamic equation of state. That inference arises as follows.⁸ Suppose that the entropy area density of any local causal horizon is $\alpha < \infty$, and that the entropy satisfies the usual thermodynamic Clausius relation

$$\delta S = \delta Q/T \quad (1)$$

for all such horizons, with δQ the (approximately defined) boost energy flux and $T = \hbar/2\pi$ the boost temperature of the vacuum mentioned above. Then the spacetime geometry *cannot be inert*: The light rays generating the horizon must focus so that the area responds to the flux of energy in just the way implied by the Einstein equation (at least at long distances). The cosmological constant is undetermined, and the value of Newton's constant given by $G = 1/4\hbar\alpha$ *which is nonzero provided the entropy density α is finite*. This implies that the entropy density $\alpha = 1/4\hbar G = 1/4L_p^2$ is $1/4$ in units of the Planck length $L_p = \sqrt{\hbar G/c^3}$, in agreement with the Bekenstein–Hawking entropy

$$S_{\text{BH}} = A/4L_p^2 \quad (2)$$

inferred long ago for black hole horizons,^{9,10} cosmological horizons,¹¹ and acceleration horizons.^{12,13}

But how could the horizon entropy ever be finite? A naive cutoff at some short distance would select a preferred reference frame, in violation of Lorentz symmetry. Since Lorentz boost symmetry lends the vacuum its thermal character, this seems a rather unlikely means to regulate the entropy. Moreover, Lorentz violation appears to wreak havoc with black hole thermodynamics, as it entails violations of the generalized second law.^{14–16} Thus, we should look elsewhere to understand the finiteness of horizon entropy.

Since the entropy can certainly be infinite in a theory with no gravity, and since the finiteness assumption implies gravity, it seems that gravity itself should somehow render the entropy finite. A natural idea, proposed before by many researchers,⁴ is that quantum gravitational fluctuations of spacetime, which are expected to be large at the Planck scale L_p , somehow cut off the entropy of UV modes approaching

the horizon at shorter distances. This yields an entropy density of order L_p^{-2} , matching the Bekenstein–Hawking entropy. We can be a little more specific about how this might work. It would violate translation invariance of the vacuum if there were any particular location at which spacetime fluctuations became large, and it would violate Lorentz invariance if there were any particular length scale at which they became large. Rather, they are large everywhere when considered in some invariant sense. The invariant that is relevant for vacuum entanglement is the proper separation between the correlated fluctuations.

The gravitationally dressed ground state satisfies the quantum analog of the initial value constraints of general relativity, the Wheeler–DeWitt equation. This equation correlates the constrained part of the gravitational field to distributions of energy. A pair of entangled fluctuations localized and separated by a proper distance ℓ have an associated quantum energy $E \sim \hbar/\ell$, and so must entail an associated metric perturbation of order $GE/\ell \sim L_p^2/\ell^2$. When $\ell < L_p$ the perturbation is large, and the causal structure of the spacetime is strongly modified. The gravitationally dressed vacuum fluctuations thus cannot be separated by a fixed horizon when they are closer than a Planck length away from each other. In effect, the entanglement is “cloaked.”

While I have argued that gravity can make the entanglement entropy finite, the argument of course does not establish that it is precisely equal to the Bekenstein–Hawking entropy. In fact, as presented so far, it does not even establish that the entropy is of order A/L_p^2 . For one thing, it does not take into account the fact that the entropy should (presumably) grow with the number N of field species in nature. Also it does not account for the running of the gravitational coupling constant G with the length scale. To include those effects, we must allow for the fact that G at scale ℓ is determined by its value at low energies G_0 and the number and nature of field species. For a theory with N fields of the same type we should replace “ G ” in the above analysis by $G(G_0, \ell, N)$. Then the condition determining the cutoff length $\ell_c(G_0, N)$ becomes

$$\ell_c^2 = \hbar G(G_0, \ell_c, N). \quad (3)$$

(Here I am assuming that the energy that determines the gravitational dressing of a correlated pair separated at scale ℓ is still $\hbar/\ell$, independent of the field species. See below for further discussion.) If the entropy grows roughly in proportion to N (aside from the N -dependence of ℓ_c), it would then be given by

$$S \sim \frac{NA}{\ell_c(G_0, N)^2}. \quad (4)$$

This would scale as the Bekenstein–Hawking entropy (2) provided the renormalization works out such that

$$\frac{N}{[\ell_c(G_0, N)]^2} \sim \frac{1}{\hbar G_0}. \quad (5)$$

It makes some sense that the reciprocal of the low energy Newton constant G_0 is proportional to N in units of G at the cutoff scale, since the N species would each contribute to the low energy effective gravitational action.^{17–19}

However, while it looks superficially satisfactory, this reasoning is rather incomplete. For one thing, the theory is strongly coupled at the cutoff scale, and it is not really clear how the number of species affects the result except perturbatively. Also, since there are N times as many independent fluctuating fields at each scale, one might naively expect an extra factor of $\sqrt{N}$ weighting the fluctuation energy, which would change the scaling of ℓ_c with N . Moreover, nonminimally coupled fields with certain sign of the nonminimal term can push the flow of the gravitational coupling in the opposite direction,^{20–22} and gauge vector²³ and graviton tensor²⁴ fields seem (at least using existing methods) to do the same. Finally, the initial condition for the renormalization flow needs to be set. A simple relation of the form (5) makes sense only if the entire low energy gravitational action is induced by the matter fluctuations.¹⁹ That could be so, but there is no apparent reason to suppose that it *must* be so.

All these complications seem to call into question the validity of the notion that entanglement entropy lies at the root of the Bekenstein–Hawking entropy. However, the thermodynamic derivation of the Einstein equation mentioned above sidesteps all of these difficulties. According to that derivation, if the horizon entanglement entropy is finite and satisfies the Clausius relation (1) (as should any entropy near equilibrium) then, whatever the underlying UV physics, it will always be equal to the Bekenstein–Hawking entropy with respect to the low energy Newton constant that appears in Einstein’s equation.

This sounds rather satisfactory, but it can be questioned from another direction. The Einstein equation is presumably just the lowest order approximation to a field equation that has higher curvature terms, and in such theories the entropy of stationary black hole horizons involves curvature corrections.^{25–27} Hence, to obtain such corrections to the equation of state, we should presumably begin with corrections to the horizon entropy function, for example constructed from the local curvature tensor and horizon geometry. However, despite many attempts^{28–34} to show it, except for the very special case where the entropy density is just a function of the spacetime Ricci scalar, the Clausius relation for an intrinsic entropy of all local causal horizons does not appear to be equivalent to a local tensor field equation. (In Ref. 34, we did manage to consistently derive some higher derivative field equations in this way, but the assumed entropy depended on arbitrary features of an approximate local Killing vector, so was not intrinsic to the spacetime and horizon.)

Should we infer from this failure that the thermodynamic derivation is just a fluke that works for Einstein gravity, but is not really fundamental? I think not. There is a good physical reason for the failure. The “heat” δQ in the Clausius relation (1) is taken as the boost energy flux, whose definition involves an approximate local boost Killing vector ξ . A flat spacetime has exact boost Killing vectors, but

in a curved spacetime ξ is defined only up to ambiguities of order $(\ell/L_{\text{curv}})^2$, where L_{curv} is the typical local radius of curvature of the spacetime in some frame and ℓ is the length over which ξ is defined. Now consider what happens if there is a curvature term $L_1^2 R$ in the entropy density, with L_1 some constant with dimensions of length and R some curvature quantity. This is suppressed relative to the area term by a factor $(L_1/L_{\text{curv}})^2$. If the ξ ambiguity is to be smaller than this we must restrict to a region of size $\ell < L_1$. If $L_1 \sim L_p$, as expected in a theory in which the Planck length is the only UV scale, this would require the region to be smaller than the Planck length. But we have seen that the quantum fluctuations of the metric associated with a pair at that scale are large, which invalidates the application of the thermodynamic Clausius relation to such a small region of a classical horizon. Thus, in a theory with only one UV length scale, we should not expect to be able to capture corrections to the horizon entropy beyond the area term by a local thermodynamic argument. (Corrections can of course be captured by global considerations involving stationary black hole configurations.)

But how about a theory in which $L_1 \gg L_p$? Then it would appear the corrections to the entropy could be larger than the ambiguity in $\delta Q/T$, so the local thermodynamic derivation should be able to capture them. However, it seems that in this case the limit to localization is no longer L_p but instead is the longer scale L_1 . For instance, in string theory, the low energy effective action has $L_1 \sim L_s$, where $L_s \gg L_p$ is the string length. When length scales smaller than L_s are probed, an infinite number of higher curvature terms in the action are equally important, and a stringy description of the degrees of freedom is necessary. One might imagine the only higher curvature term in the action is the R^2 term, but this requires unnatural fine tuning, and there is probably no physical theory that actually behaves this way.

The simplicity of the area entanglement entropy belies a microscopic complexity. Yet, without knowing in detail how to identify and count the precise degrees of freedom, thermodynamic reasoning allows us to deduce the universal relation between horizon entropy and Einstein gravity. But thermodynamic reasoning has its limitations; in particular, it seems we should not expect to understand corrections to Einstein gravity in this way.

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