

General Conditions for Approximate Quantum Error Correction and Near-Optimal Recovery Channels

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We derive necessary and sufficient conditions for the approximate correctability of a quantum code, generalizing the Knill-Laflamme conditions for exact error correction. Our measure of success of the recovery operation is the worst-case entanglement fidelity. We show that the optimal recovery fidelity can be predicted exactly from a dual optimization problem on the environment causing the noise. We use this result to obtain an estimate of the optimal recovery fidelity as well as a way of constructing a class of near-optimal recovery channels that work within twice the minimal error. In addition to standard subspace codes, our results hold for subsystem codes and hybrid quantum-classical codes.

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Introduction.—Given the extreme fragility of quantum coherence, quantum error-correction procedures are believed to be essential for the successful implementation of quantum communication or computation. Exact correctability is characterized in general terms by the Knill-Laflamme (KL) conditions [1] which specify the set of correctable errors for a particular code. However, practically useful codes need not be exactly correctable for any given noise model. In fact, a few exceptional examples show that allowing for a negligible error in the recovery can lead to surprisingly better codes [2,3]. This indicates that assuming exact correctability is too strong a restriction. It is therefore of considerable interest to find appropriately weaker error-correction conditions.

In this Letter, we generalize the KL conditions to the case of approximate correctability. We view the KL conditions as a statement about the information gathered by the environment causing the noise [4] (a fact previously noted in Ref. [5]). Thus, our analysis makes essential use of the concept of *complementary* channel [6]. Together with tools introduced in Ref. [7], this provides the basis for our main technical result (Theorem 1), which we use to obtain easily computable estimates of the optimal recovery error. We also propose a class of near-optimal recovery channels, which offers a significant simplification to the problem of finding an optimal recovery operation [8].

The analysis of approximate error correction depends on the figure of merit used to compare the states after correction to the input states. In this work, we focus on the entanglement fidelity minimized over all input states (also known as worst-case entanglement fidelity). The entanglement fidelity [9] has been shown to be the pertinent fidelity measure in both quantum communication and computation scenarios since it estimates not only how well the state of the system under correction is preserved but also how its entanglement with auxiliary systems is maintained. Minimization over all inputs is essential if one is interested in guaranteeing a given fidelity when the state to

be corrected is not known, as in the case of quantum computing. In contrast, most previous work has considered input-dependent fidelities [10,11]. Sufficient conditions for approximate correctability under the worst-case entanglement fidelity were proposed in Ref. [12]. Here, we obtain both sufficient and necessary conditions which are a direct generalization of the KL conditions. Moreover, we prove our result in a very general context, namely, for the approximation of any channel, not necessarily the identity map on the code. One advantage of this generality is that our results apply directly to the more general schemes of subsystem, or operator quantum error correction [13–16]. The present results are also strictly stronger than those of Refs. [4,17] which are based on the diamond-norm distance rather than the fidelity.

Background.—The problem of quantum error correction can be formulated as follows: we are given a channel $\mathcal{N}$ which can represent either a communication channel or the open dynamics of a physical system which we would like to use as a quantum memory. The goal is to find an encoding operation $\mathcal{E}$ and a decoding (or recovery) operation $\mathcal{R}$, such that the full operation $\mathcal{R}\mathcal{N}\mathcal{E}$ is equal to the identity map. One usually assumes that the encoding is of the form $\mathcal{E}(\rho) = V\rho V^\dagger$ where V is an isometry embedding a small Hilbert space (the code) into the larger physical Hilbert space on which $\mathcal{N}$ acts.

Given $\mathcal{N}$ and $\mathcal{E}$, the KL conditions [1] provide a simple way of testing whether a recovery channel $\mathcal{R}$ exists. In addition, these conditions help reasoning about error correction. For instance, one can use them together with the no-cloning theorem to easily demonstrate that it is not possible to encode a qubit in n qubits and faithfully decode it if $n/4$ or more arbitrary qubit errors occur. The reason we mention this particular example is that it is known to fail dramatically if we allow for an arbitrarily small reconstruction error (provided n is large enough). Indeed, it was shown in Ref. [3] that one can encode quantum information in n qubits undergoing almost $n/2$

convex sets. Hence, we obtain $\max_{\mathcal{R}} F(\mathcal{R}\mathcal{N}, \mathcal{M}) = \max_{A'} \max_A \min_{\rho} \operatorname{Re} g_{\rho}(A, A') = \max_{\mathcal{R}'} F(\hat{\mathcal{N}}, \mathcal{R}'\hat{\mathcal{M}})$, where $\|A'\|, \|A\| \leq 1$. ■

Note that Eq. (3) can be seen as a necessary and sufficient condition for approximate correctability: for a given $\delta \in [0, 1]$, there exists a channel $\mathcal{R}$ such that $F(\mathcal{R}\mathcal{N}, \mathcal{M}) = 1 - \delta$, if and only if there exists a channel $\mathcal{R}'$ such that $F(\hat{\mathcal{N}}, \mathcal{R}'\hat{\mathcal{M}}) = 1 - \delta$. We will see that for a large class of problems of interest, the existence of $\mathcal{R}'$ is much easier to establish than that of $\mathcal{R}$.

Knill-Laflamme conditions.—Consider the case $\mathcal{M} = \text{id}$. An example of a channel complementary to the identity is the trace: $\hat{\mathcal{M}} = \text{Tr}$, whose target is one-dimensional. A channel whose source is one-dimensional outputs a single state. Hence, $\mathcal{R}'\hat{\mathcal{M}}(\rho) = \rho_0$, $\forall \rho$, where ρ_0 is a fixed state. Theorem 1 thus says that $\max_{\mathcal{R}} F(\mathcal{R}\mathcal{N}, \text{id}) = 1$ if and only if $\hat{\mathcal{N}}(\rho) = \rho_0 \text{Tr}(\rho)$, $\forall \rho$. Explicitly, suppose that the channel $\mathcal{N}$ consists of an encoding specified by an isometry V followed by noise with channel elements E_i . In terms of matrix components, the condition that $\hat{\mathcal{N}}$ be a constant channel with output ρ_0 reads $V^\dagger E_i^\dagger E_j V = \lambda_{ij} \mathbf{1}$, where $\lambda_{ij} = \langle i|\rho_0|j\rangle$. One obtains the most familiar form of the KL conditions by expressing these equations using the projector $P = VV^\dagger$ on the code: $PE_i^\dagger E_j P = \lambda_{ij} P$.

More generally, if $\mathcal{M} = \mathcal{P}_{\mathcal{A}}$ projects on an algebra $\mathcal{A}$, then we obtain the general correctability conditions for an algebra [16], namely, $[A, V^\dagger E_i^\dagger E_j V] = 0$ for all $A \in \mathcal{A}$. This can be shown by noting that $\hat{\mathcal{P}}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}'}$, where $\mathcal{A}'$ is the commutant of $\mathcal{A}$, i.e., the set of operators commuting with all $A \in \mathcal{A}$ (see [4] for more details.) In particular, when the algebra $\mathcal{A}$ consists of all operators acting on a subsystem, this yields the conditions for the correctability of a subsystem code [14].

Let us show explicitly how Theorem 1 can be understood as a perturbation of the KL conditions in the case $\mathcal{M} = \text{id}$. Since later in our analysis we will use triangle inequalities, it is convenient to measure the error of imperfect recovery by a fidelity-based distance function. We will consider the Bures distance [23] based on the entanglement fidelity, $d_{\rho}(\mathcal{N}, \mathcal{M}) = \sqrt{1 - F_{\rho}(\mathcal{N}, \mathcal{M})}$. Note that

$$d(\mathcal{N}, \mathcal{M}) := \max_{\rho} d_{\rho}(\mathcal{N}, \mathcal{M}) = \sqrt{1 - F(\mathcal{N}, \mathcal{M})} \quad (5)$$

satisfies the triangle inequality: $d(\mathcal{N}, \mathcal{M}) = \max_{\rho} d_{\rho}(\mathcal{N}, \mathcal{M}) \leq \max_{\rho} [d_{\rho}(\mathcal{N}, \mathcal{R}) + d_{\rho}(\mathcal{R}, \mathcal{M})] \leq d(\mathcal{N}, \mathcal{R}) + d(\mathcal{R}, \mathcal{M})$.

Definition 1.—We will say that a code characterized by the encoding map $\mathcal{E}$ is ε -correctable under the noise channel $\mathcal{N}$, if there exists a recovery channel $\mathcal{R}$ such that $d(\mathcal{R}\mathcal{N}\mathcal{E}, \text{id}) \leq \varepsilon$.

Corollary 2.—A code defined by the projector P is ε -correctable under a noise channel $\mathcal{N}$, if and only if $PE_i^\dagger E_j P = \lambda_{ij} P + PB_{ij}P$, where λ_{ij} are the components of a density operator, and $d(\Lambda + \mathcal{B}, \Lambda) \leq \varepsilon$ where

$$\Lambda(\rho) = \sum_{ij} \lambda_{ij} \text{Tr}(\rho) |i\rangle\langle j| \quad \text{and} \quad (\Lambda + \mathcal{B})(\rho) = \Lambda(\rho) + \sum_{ij} \text{Tr}(\rho B_{ij}) |i\rangle\langle j|.$$

Proof.—Let us denote the encoding channel by $\mathcal{E}$. It is clear from Theorem 1 that the code is ε -correctable if and only if there exists a state ρ_0 such that $d(\hat{\mathcal{N}}\mathcal{E}, \Lambda) \leq \varepsilon$, where Λ is defined as in the statement of the corollary with $\lambda_{ij} = \langle i|\rho_0|j\rangle$. Also, from Eq. (1), we see that indeed $\hat{\mathcal{N}}\mathcal{E} = \Lambda + \mathcal{B}$ since the operators $V^\dagger B_{ij} V$ are defined by $PB_{ij}P = PE_i^\dagger E_j P - \lambda_{ij} P$. ■

It is not *a priori* clear how useful this condition can be since it does not specify how to find an optimal set of coefficients λ_{ij} . We will now show, in a more general setting, that we can find a whole set of explicit guesses for λ_{ij} which are guaranteed to yield a value of ε which is less than twice the optimal one. Explicitly, this is the case whenever $\lambda_{ij} = \text{Tr}(\sigma E_i^\dagger E_j)$ for some state σ .

Near-optimal correction.—We saw that in the exact case (fidelity one), Theorem 1 yields the necessary and sufficient conditions for all quantum error-correction schemes when $\mathcal{M}$ projects on an algebra. Here, we want to show that it also yields useful conditions for the approximate version of these schemes. The problem is that in general it may not be easier to compute $\max_{\mathcal{R}'} F(\hat{\mathcal{N}}, \mathcal{R}'\hat{\mathcal{M}})$ than $\max_{\mathcal{R}} F(\mathcal{R}\mathcal{N}, \mathcal{M})$. However, we will show that when $\hat{\mathcal{M}}$ is a projection, i.e., it satisfies $\hat{\mathcal{M}}^2 = \hat{\mathcal{M}}$ (which is the case for error correction), we can guess a whole class of channels $\hat{\mathcal{R}}'$ for which $F(\hat{\mathcal{N}}, \hat{\mathcal{R}}'\hat{\mathcal{M}})$ yields a good approximation to the optimal worst-case fidelity $\max_{\mathcal{R}} F(\mathcal{R}\mathcal{N}, \mathcal{M})$. Moreover, we can build the corresponding near-optimal recovery channels $\hat{\mathcal{R}}$.

Corollary 3.—Suppose that $\hat{\mathcal{M}}^2 = \hat{\mathcal{M}}$. Then, $\frac{1}{2}d(\hat{\mathcal{N}}, \hat{\mathcal{N}}\hat{\mathcal{M}}) \leq \min_{\mathcal{R}} d(\mathcal{R}\mathcal{N}, \mathcal{M}) \leq d(\hat{\mathcal{N}}, \hat{\mathcal{N}}\hat{\mathcal{M}})$.

Proof.—First note that Theorem 1 expressed in terms of d reads $\varepsilon_0 := \min_{\mathcal{R}} d(\mathcal{R}\mathcal{N}, \mathcal{M}) = \min_{\mathcal{R}} d(\hat{\mathcal{N}}, \mathcal{R}'\hat{\mathcal{M}})$. The rightmost inequality follows from picking the non-optimal $\mathcal{R}' = \hat{\mathcal{N}}$. For the leftmost inequality, suppose that $\mathcal{R}'_0$ minimizes $d(\hat{\mathcal{N}}, \mathcal{R}'\hat{\mathcal{M}})$. Then using the triangle inequality, $d(\hat{\mathcal{N}}, \hat{\mathcal{N}}\hat{\mathcal{M}}) \leq d(\hat{\mathcal{N}}, \mathcal{R}'_0\hat{\mathcal{M}}) + d(\mathcal{R}'_0\hat{\mathcal{M}}, \hat{\mathcal{N}}\hat{\mathcal{M}})$. We know that the first term is equal to ε_0 since $\mathcal{R}'_0$ is optimal. For the second term, note that $d(\mathcal{R}'_0\hat{\mathcal{M}}, \hat{\mathcal{N}}\hat{\mathcal{M}}) = d(\mathcal{R}'_0\hat{\mathcal{M}}^2, \hat{\mathcal{N}}\hat{\mathcal{M}}) \leq d(\mathcal{R}'_0\hat{\mathcal{M}}, \hat{\mathcal{N}}) = \varepsilon_0$. Hence, $d(\hat{\mathcal{N}}, \hat{\mathcal{N}}\hat{\mathcal{M}}) \leq 2\varepsilon_0$. ■

Note that computing $d(\hat{\mathcal{N}}, \hat{\mathcal{N}}\hat{\mathcal{M}})$ requires a convex maximization over inputs only [19], which is a significant simplification over the minimax $\min_{\mathcal{R}} d(\mathcal{R}\mathcal{N}, \mathcal{M})$.

Near-optimal recovery channel.—Let us show how we can construct a recovery channel $\hat{\mathcal{R}}$ which performs as well as guaranteed by our bounds, i.e.,

$$d(\hat{\mathcal{R}}\mathcal{N}, \mathcal{M}) \leq d(\hat{\mathcal{N}}, \hat{\mathcal{N}}\hat{\mathcal{M}}). \quad (6)$$

We first need to find a saddle point (ρ_0, A_0) of $\operatorname{Re} g_{\rho}(A, U')$, where U' yields $\hat{\mathcal{N}}$ through $\hat{\mathcal{N}}(\rho) = \text{Tr}_2[U'(\rho \otimes |0\rangle\langle 0|)(U')^\dagger]$. This implies that

$F(\hat{\mathcal{N}}, \hat{\mathcal{N}} \hat{\mathcal{M}}) = \text{Re } g_{\rho_0}(A_0, U')$. For instance, in the case $\mathcal{M} = \text{id}$, one may first find a ρ_0 that minimizes $\text{Tr}(|X_{\rho_0}|)$, which is a convex optimization problem [19]. If ρ_0 is full rank and unique, then A_0 can be chosen to be any unitary obtained from the polar decomposition of X_{ρ_0} . Generally, the saddle-point operator A_0 yields a channel $\mathcal{S}(\rho) := \text{Tr}_2(A_0(\rho \otimes |0\rangle\langle 0|)A_0^\dagger)$ which may be trace decreasing but can always be completed to a trace-preserving channel $\tilde{\mathcal{R}}(\rho) = \mathcal{S}(\rho) + \text{Tr}[\rho - \mathcal{S}(\rho)]\tau$ for any fixed state τ . $\tilde{\mathcal{R}}$ then satisfies Eq. (6). Indeed, $F(\hat{\mathcal{N}}, \hat{\mathcal{N}} \hat{\mathcal{M}}) = \min_{\rho} \text{Re } g_{\rho}(A_0, U') \leq \max_{A', \|A'\| \leq 1} \min_{\rho} \text{Re } g_{\rho}(A_0, A') = \min_{\rho} \max_{U'} \text{Re } g_{\rho}(A_0, U') \leq F(\tilde{\mathcal{R}} \mathcal{N}, \mathcal{M})$.

Example.—In the standard case $\mathcal{M} = \text{id}$, the estimate $\delta(\mathcal{N}) = d(\hat{\mathcal{N}}, \hat{\mathcal{N}} \hat{\mathcal{M}})$ from Corollary 3 is given explicitly in terms of the fidelity by

$$F(\hat{\mathcal{N}}, \hat{\mathcal{N}} \hat{\mathcal{M}}) = \min_{\rho} \text{Tr} \sqrt{\sum_{ij} E_i \rho^2 E_j^\dagger \text{Tr}(E_j \sigma E_i^\dagger)},$$

where σ is an arbitrary state resulting from the freedom in choosing the complementary channel $\hat{\mathcal{M}}$. As an example, we applied our results to the code proposed in Ref. [2], for which we numerically computed $\delta(\mathcal{N}\mathcal{E})$, where $\mathcal{E}$ is the encoding, and $\mathcal{N}_\gamma$ is the noise model with strength γ considered there [24]. Our calculations show that this code is good in terms of the worst-case entanglement fidelity, in the sense that it yields a fidelity distance of order γ instead of the uncorrected order $\sqrt{\gamma}$. The necessity of our criterion shows, in particular, that one cannot obtain a better asymptotic behavior with this code.

Fixed-state fidelity.—We finally note that our method can also be applied to the problem of error correction on a fixed input state (a scenario studied, e.g., in Ref. [10]). Indeed, for any state ρ we also have $\max_{\mathcal{R}} F_{\rho}(\mathcal{R} \mathcal{N}, \mathcal{M}) = \max_{\mathcal{R}'} F_{\rho}(\hat{\mathcal{N}}, \mathcal{R}' \hat{\mathcal{M}})$. The argument is simpler as no minimization over ρ is involved. In the case $\mathcal{M} = \text{id}$, and using a reasoning very similar to the one we used for the worst-case fidelity, we can obtain the estimate $\frac{1}{2} d_{\rho}(\hat{\mathcal{N}}, \mathcal{S}) \leq \min_{\mathcal{R}} d_{\rho}(\mathcal{R} \mathcal{N}, \text{id}) \leq d_{\rho}(\hat{\mathcal{N}}, \mathcal{S})$ where $\mathcal{S}(\sigma) := \hat{\mathcal{N}}(\rho) \text{Tr}(\sigma)$. In addition, the corresponding near-optimal recovery channel can be built using the same method as for the worst-case fidelity with the simplification that no search of a saddle point is involved.

Conclusion.—In summary, we have generalized the Knill-Laflamme conditions to the case of approximate correctability, including standard codes, subsystem codes, and hybrid quantum-classical codes. We obtained easy-to-calculate estimates of the optimal recovery error and proposed a class of near-optimal recovery channels for the worst-case entanglement fidelity that work within twice of the optimal error. These results provide a framework for studying error correction under general noise models and allow for a significant simplification to the task of finding optimal error-correction procedures, thus offering a prom-

ising tool to efficiently address the problem of decoherence control in realistic scenarios.

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