PNAS RESEARCH ARTICLE | PHYSICS

Unveiling order from chaos by approximate 2-localization of random matrices

Nicolas Loizeau[a], Flaviano Morone[a], and Dries Sels[a,b,1]

Edited by Edouard Brézin, École normale supérieure, Paris, France; received May 12, 2023; accepted August 21, 2023

Quantum many-body systems are typically endowed with a tensor product structure. A structure they inherited from probability theory, where the probability of two independent events is the product of the probabilities. The tensor product structure of a Hamiltonian thus gives a natural decomposition of the system into independent smaller subsystems. It is interesting to understand whether a given Hamiltonian is compatible with some particular tensor product structure. In particular, we ask, is there a basis in which an arbitrary Hamiltonian has a 2-local form, i.e., it contains only pairwise interactions? Here we show, using analytical and numerical calculations, that a generic Hamiltonian (e.g., a large random matrix) can be approximately written as a linear combination of two-body interaction terms with high precision; that is, the Hamiltonian is 2-local in a carefully chosen basis. Moreover, we show that these Hamiltonians are not fine-tuned, meaning that the spectrum is robust against perturbations of the coupling constants. Finally, by analyzing the adjacency structure of the couplings J_{ij}, we suggest a possible mechanism for the emergence of geometric locality from quantum chaos.

random matrices | quantum chaos | locality | entanglement

Significance

The classical world is local and most classical Hamiltonians (e.g., Newtonian gravity) can be written as a sum of 2-body interactions. However, quantum mechanics allows for more general nonlocal interactions. If the classical realm emerges from the quantum one, one needs to explain why we see the world as local, i.e., explain how locality can emerge from quantum mechanics itself. We show that it is a matter of perspective: Consider a very generic quantum system with many-body interactions, it is very likely that there is a way to represent it as a local system with few-body interactions.

Typically, to obtain a quantum description of the dynamics of a system we go through a procedure of canonical quantization, or as Dirac described it (1), we work by classical analogy. While this procedure has proven extremely powerful, it is also profoundly unsatisfying. How can it be that, in order to describe the microscopic fundamental quantum laws, we first need to know the corresponding classical Hamiltonian that governs the behavior of the system? Aren't classical mechanics supposed to emerge out of quantum mechanics? More concretely, since most classical Hamiltonians have a rather simple form, it raises the question of how much we have constrained quantum mechanics by this.

To set the stage of our discussion, we start by elucidating some very basic concepts (2–4). Quantum mechanics, in and of itself, is independent of one's choice of basis, i.e., it is invariant under unitary transformations. In particular, it is invariant under time evolution, which is a unitary transformation of the state of the system. This puts a constraint on the set of observables that can actually be measured. The absence of such constraint would immediately render time an irrelevant concept, e.g., instead of measuring observable O one can just measure $\exp(-iHt)O\exp(iHt)$ to travel backward for time t. Therefore, any discussion should be restricted to a specific set of observables.

In practice, the set of observables we have access to in our universe is very limited, and dictated by experimental constraints. Empirically, there is close connection between the Hamiltonian and the observables that are accessible; for example, in quantum field theory, both have simple algebraic expressions in terms of creation and annihilation operators (5). Simply put, we write down the Hamiltonian having already in mind the observables we are going to measure (6). It is this tacit assumption of a simple relation between the kinematics of the system and the accessible observables that we wish to investigate in this work.

To put it slightly differently, in one of his seminal papers on quantum mechanics Schrödinger called entanglement the characteristic trait of quantum mechanics that enforces one to depart from classical thinking (7). Entanglement, however, is a basis dependent concept. It requires one to specify the objects that naturally appear as independent, i.e., disentangled. A priori it is not clear what such independent classical objects should be (8–10): Why should one basis be more natural than the other?

Given a Hamiltonian, the only piece of intrinsic, basis invariant, information is its spectrum. Hence, all systems with the same energy spectrum are equivalent. The only difference thus hides in how we gain access to local physical quantities in those systems. This leads to the following question: Given a Hamiltonian, is it possible to find a basis

Author affiliations: [a]Department of Physics, New York University, New York, NY 10003; and [b]Center for Computational Quantum Physics, Flatiron Institute, New York, NY 10010

Author contributions: D.S. designed research; N.L., F.M., and D.S. performed research; N.L. and D.S. analyzed data; and N.L., F.M., and D.S. wrote the paper.

The authors declare no competing interest.

This article is a PNAS Direct Submission.

[1]To whom correspondence may be addressed. Email: dsels@nyu.edu.

This article contains supporting information online at https://www.pnas.org/lookup/suppl/doi:10.1073/pnas.2308006120/-/DCSupplemental.

Published September 19, 2023.

in which it has a simple tensor product form, such as a linear combination of only two-body interaction terms? And if so, can it be made geometrically local? The problem has recently been considered by Cotler et al. (11), who used a simple counting argument to show that it is not possible in general. However, the question of to what precision it can be done remains open and is the subject of this work. We numerically explore a view in which local preferred basis emerges from quantum chaos (12, 13) by looking for bases in which random matrices can be approximately written as 2-local Hamiltonians. A related, although quite different, idea has recently been put forward by Freedman and Zini (14, 15) who argue for a novel mechanism of spontaneous symmetry breaking acting on the level of the probability distribution of Hamiltonians rather than on the level of quantum states.

1. 2-localization

Consider a generic Hamiltonian H that acts on a Hilbert space $\mathcal{H}$. For simplicity let us restrict ourselves to $\mathcal{H} = \mathbb{C}^M$, where M is taken to be a power of two. To be concrete, think of H as a random matrix drawn from the GOE ensemble (16–18). In addition, consider the set of Pauli strings $\mathsf{P}_N = \{\tau\}$, composed out of tensor products of Pauli operators σ_i^α acting on N spins, or qubits, e.g., $\tau = \sigma_1^x \otimes \sigma_2^z \otimes \cdots \otimes \mathbb{1}_N$. The set of Pauli strings P_N forms a complete basis, hence any Hamiltonian (on $\mathcal{H} = \mathbb{C}^{2^N}$) can be written as a linear combination of Pauli strings $H = \sum_{\tau\in\mathsf{P}_N} h_\tau \tau$. While generic operators are supported on all strings, there is a natural ordering in the set of Pauli strings given by their length, i.e., the number of nonidentity operators in the tensor product. Let us denote the set of all strings up to length k as P_N^k. In this work, we are particularly interested in operators that are localized on the set P_N^2 of strings of at most length two. We call a Hamiltonian 2-localizable if, after some carefully chosen unitary transformation U, it is entirely supported on P_N^2 (Fig. 1*A*), i.e., there exists a set of couplings $\{h_\tau\}$ such that:

$$UHU^\dagger = \sum_{\tau\in\mathsf{P}_N^2} h_\tau \tau \equiv \sum_{ij,\alpha\beta} J_{ij}^{\alpha\beta}\sigma_i^\alpha \otimes \sigma_j^\beta + \sum_{i,\alpha} h_i^\alpha \sigma_i^\alpha, \qquad [1]$$

where σ_i^α is the α-Pauli matrix acting on the i'th qubit.

For example, let us consider a three-qubit problem with a Hamiltonian $H = XXX$, where X stands for σ^x and the tensor product symbol is dropped for clarity. This Hamiltonian is 3-local and its spectrum contains an equal number of $+1$ and -1 eigenvalues. But so does the spectrum of $H' = X\mathbb{1}Z$, which is, instead, a 2-local Hamiltonian. Therefore there must exist a unitary U that brings H into H' and thus 2-localizes the problem. In fact, it easy to show that $U = \frac{\sqrt{2}}{2}(\mathbb{1}\mathbb{1}\mathbb{1} + i\mathbb{1}XY)$ does the job, i.e.,

$$UXXXU^\dagger = X\mathbb{1}Z. \qquad [2]$$

We can say even more: the Hamiltonian $H'' = X\mathbb{1}\mathbb{1}$ is 1-local and isospectral to H; hence, in this simple example, H can be 1-localized. In general, the solution is not unique, as one can permute all the spins and apply arbitrary single-spin rotations.

A necessary condition for exact 2-localization of an arbitrary matrix H is that there are enough degrees of freedom in the local subspace to encode the eigenvalues of H (11) (Fig. 1*B*). There are $3N + \frac{9}{2}N(N-1)$ allowed strings if H is complex (i.e., drawn from GUE) and $2N + \frac{5}{2}N(N-1)$ if H is real (i.e., drawn from GOE); therefore, the above condition is satisfied for $N \leq 8$ in the complex case and $N \leq 6$ in the real case, which is consistent with GOE matrices being numerically localizable for $N \leq 6$ as we show in the next section. Before moving on to the main point of the paper, let us note the previous argument has two caveats: first, it does not say anything about how close one can approximate an operator by a two-local one; second, it does not imply that all operators $N \leq 6$ can be 2-localized (it only implies that not all Hamiltonians can be 2-localized when $N > 6$). To illustrate the latter, we argue that low-rank projectors cannot be localized, even in small systems, as we prove next. If a rank-K projector is 2-localizable, then there exists a rank-K projector that is 2-local. So let us derive a bound on the rank of a 2-local projector P. P is 2-local iff $P = \sum_{\tau\in\mathsf{P}_N^2} h_\tau \tau$. The rank of P is $K = \mathrm{Tr}(P) = \mathrm{Tr}(P^2) = \sum_{\tau\in\mathsf{P}_N^2} h_\tau \mathrm{Tr}(\tau P)$. Note that $\mathrm{Tr}(\tau P) = 2^N h_\tau$, so $K = \frac{1}{2^N}\sum_{\tau\in\mathsf{P}_N^2} \mathrm{Tr}(\tau P)^2$. Moreover, $P = \sum_q |q\rangle\langle q|$, so we have

$$K = \frac{1}{2^N}\sum_{\tau\in\mathsf{P}_N^2}\left(\sum_q \langle q|\,\tau\,|q\rangle\right)^2 \leq \frac{1}{2^N}\mathcal{N}_2 K^2 \rightarrow K \geq \frac{2^N}{\mathcal{N}_2}, \qquad [3]$$

where $\mathcal{N}_2 = O(N^2)$ is the number of Pauli strings of length 2. So any 2-localizable projector on $\mathbb{C}^{2^N}$ has rank greater than

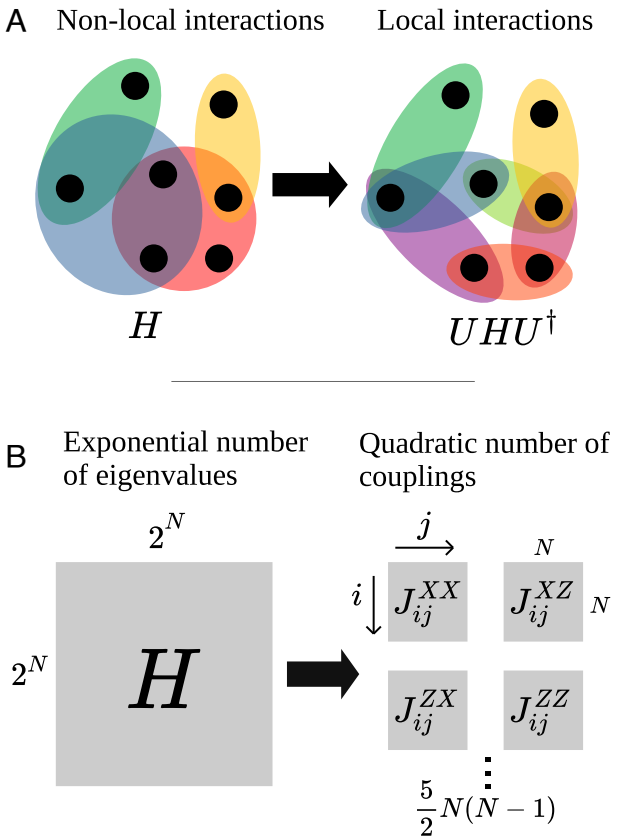

Fig. 1. (*A*) A generic Hamiltonian H comprises many body interactions. After 2-localization via a unitary U, the Hamiltonian $UHU^\dagger$ is represented in a new basis such that it only contains 2-body interactions. (*B*) A N qubit quantum Hamiltonian is a $2^N \times 2^N$ Hermitian matrix and has 2^N intrinsic degrees of freedom (its eigenvalues). After 2-localization, these 2^N degrees of freedom are compressed into $O(N^2)$ couplings J_{ij} and $O(N)$ fields h_i which specify the 2-local Hamiltonian.

$O(2^N/N^2)$, or, equivalently, low-rank projectors cannot be 2-local. This simply expresses the intuition that one needs nonlocal information to express a low entropy state ρ.

2. Method

There exists a unitary U that localizes a Hamiltonian H if and only if there exists a local Hamiltonian H' that has the same spectrum as H. One can localize H by looking for a local Hamiltonian with the same spectrum. Let us define the cost function

$$C = \frac{1}{2^{N+1}} \sum_{i=1}^{2^N} (E_i - \mathcal{E}_i)^2, \qquad [4]$$

where E_i are the eigenvalues of H and $\mathcal{E}_i \equiv \mathcal{E}_i(h)$ are the eigenvalues of a local Hamiltonian $H' = \sum_{\tau \in \mathrm{P}_N^2} h_\tau \tau$. The cost function C measures the mean squared localization error, that is how close the spectrum of the 2-local Hamiltonian H' is to the spectrum of the original Hamiltonian H.

Localizing H is equivalent to finding coefficients h_τ that minimize C. Note that the gradient of C is

$$\frac{\partial C}{\partial h_\tau} = h_\tau - \frac{1}{2^N} \sum_n E_n \langle n | \tau | n \rangle, \qquad [5]$$

where $|n\rangle \equiv |n(h)\rangle$ is the eigenvector of H' with eigenvalue $\mathcal{E}_n$, as we show explicitly in *SI Appendix*, section 3. In practice, we minimize C using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) gradient descent method (19–23). In the general case, where the τ includes X, Y, and Z, we can 2-localize random matrices H from the GOE ensemble up to $N = 14$ (i.e., matrices of maximum size $2^{14} \times 2^{14}$). The main bottleneck is the time required to perform the diagonalization of H' at every step. However, in the case where the τ are products of Z's only (diagonal case) we can 2-localize GOE matrices up to $N = 20$. When $N > 16$ we use a tridiagonal Hermite matrix ensemble to generate the initial GOE spectrum (24).

3. Results

We generate sets of 200 N-qubits random Hamiltonians from the Gaussian orthogonal ensemble (GOE); then attempt to 2-localize them. For $N \leq 6$ every particular Hamiltonian H was localizable up to machine precision. For $N > 6$, the larger the system, the better they can be localized and the error decreases exponentially with N as shown in Fig. 2, i.e., in the accessible regime the error goes down faster than the inverse dimension of Hilbert space 2^{-N}.

Since we retrieve the spectrum with exponential precision it seems likely we do not just retrieve coarse-grained information about the density of states of H, but reproduce all essential features. To verify this, we compare the spectral form factor (SFF) of the retrieved ensemble of 2-local H' with that of the GOE ensemble. The SFF can be thought of as the Fourier transform of the two-point correlation function of the spectrum, i.e., it measures how fluctuations in the density of states are correlated

$$\mathrm{SFF}(t) = \left\langle |Z(H, it)|^2 \right\rangle, \qquad [6]$$

where $Z(H, it)$ denotes the generating function

$$Z(H, it) = \mathrm{Tr}\, e^{itH}, \qquad [7]$$

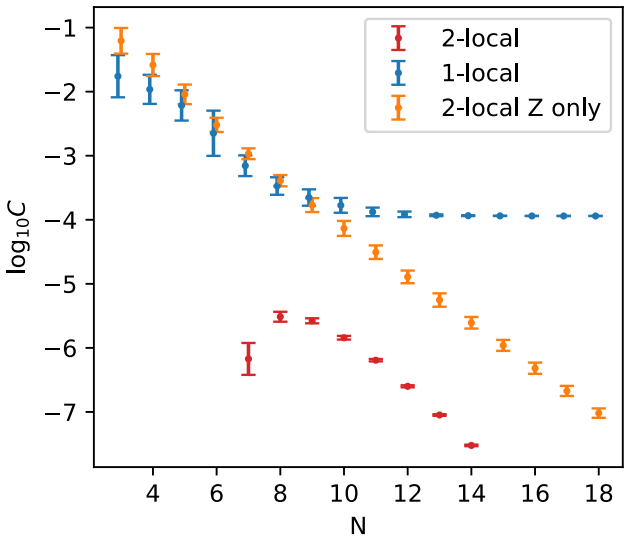

Fig. 2. Localization error C defined in Eq. **4** versus system size N. Red data points—labeled "2 local"—are errors on the localization of a GOE spectrum with a 2-local Hamiltonian of the general form given by Eq. **1**. For $N \leq 6$ the error is below machine precision, indicating exact 2-localization, in agreement with the simple counting argument given in the main text. For $N \geq 10$ the localization error vanishes as $2^{-\alpha N}$, faster than the inverse of the Hilbert space dimension. The best fit to the cost gives $\alpha = 1.4$. Blue symbols ("1 local") are localization errors obtained by using a 1-local Hamiltonian $H' = \sum_i h_i \sigma_i^z$. The latter can be chosen diagonal, since local rotations do not change the k-locality. In this case, the localization error saturates at $N = 10$ and stays constant for larger system size, implying the impossibility to 1-localize a GOE spectrum. Orange points ("2 local Z only") corresponds to the case of a 2-local Hamiltonian $H' = \sum_{ij} J_{ij} \sigma_i^z \sigma_j^z + \sum_i h_i \sigma_i^z$, describing a classical Ising model. In this case, the localization error decreases exponentially with system size ($C \sim 2^{-1.21N}$), and thus 2-localization is possible in this case. Each data point is averaged over 200 GOE matrices. Similar figures using the 1-norm and the spectral norm are shown in *SI Appendix*, Fig. S8.

and the $\langle \cdot \rangle$ refers to the ensemble average over H. The spectral form factor has a universal ramp structure at late times which is a hallmark of quantum chaos (13, 25) and Fig. 3. In typical chaotic systems, there is some nonuniversal initial behavior ending in the so-called correlation hole, the time duration of which is sometimes called the Thouless time. After the Thouless time follows a universal ramp which stops at the Heisenberg time. The study of this universal behavior has yielded important insights into ergodicity breaking, in particular in the context of disordered many-body systems (26, 27) and SYK models (28, 29). As shown

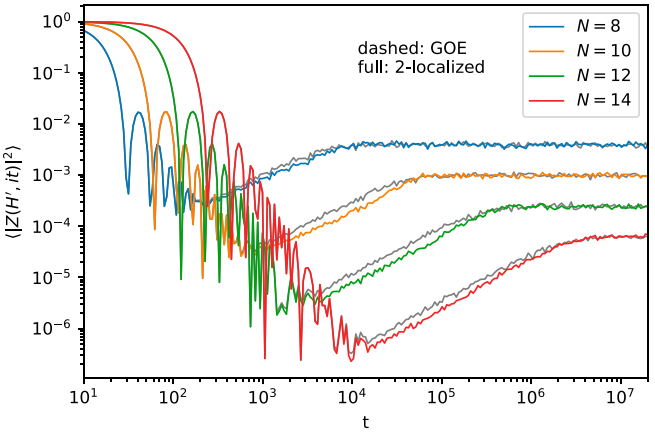

Fig. 3. Spectral form factor SFF(t) of the localized Hamiltonians (color) defined by Eq. **6**. The SFF of the initial GOE spectrum is plotted in gray for reference. Results are averaged over 200 realizations of GOE initial spectra.

in Fig. 3, we recover all essential features of the SFF in the 2-localized ensemble at all timescales. We only observe a small deviation in the ramp which can be interpreted as a small delay in the Thouless time. As expected from these SFF results, the behavior of simple observables such as 2-spin correlators is also left unchanged (*SI Appendix*, Fig. S7).

A. Stability. Having established that there are 2-local Hamiltonians that approximate a GOE matrix with exponential precision, it becomes important to understand the stability of these solutions. If small changes in the coupling constants h_τ result in a completely different spectrum this would make the 2-local Hamiltonians rather fine-tuned. Consider the Hessian of the cost function $C(h)$ in the minimum $h = h^0$ (see *SI Appendix*, section 3 for details):

$$g_{\tau\eta}(h_0) = \frac{\partial^2 C}{\partial h_\tau \partial h_\eta}\Big|_{h=h^0} = \frac{1}{2^N}\sum_n \langle n|\,\tau\,|n\rangle\,\langle n|\,\eta\,|n\rangle\,, \qquad [8]$$

where $|n\rangle \equiv |n(h)\rangle$ is the eigenvector of H'. Note that the Hessian only depends on the diagonal expectation values of 2-local operators, which are expected to behave completely differently in integrable and chaotic systems (30). In that regard, consider a 2-local H' in which the Pauli strings are restricted to commute, e.g., strings composed of only Z's, which are all diagonal. All eigenvectors of H' are eigenvectors of τ, such that the sum in expression Eq. **8** becomes a trace. Since all strings are trace orthogonal, one finds $g_{\tau\eta} = \delta_{\tau\eta}$. As a consequence, for commutative 2-local Hamiltonians, a small change in the coupling constants Δh results in a change of the cost function $\Delta C \approx \|\Delta h\|^2$. Since the coupling constants themselves are $O(1/N)$ this requires exponential precision in the specification of the coupling constants h to maintain the exponential decrease in the cost function seen in Fig. 2. Also, since the metric $g_{\tau\eta}$ becomes diagonal, there are no particular directions of stability: The system is equally susceptible to small perturbations in all directions.

The situation should be different for generic H' in which the diagonal expectation values in expression Eq. **8** are expected to obey the eigenstate thermalization hypothesis (ETH) (30). According to ETH, expectation values of (local) observables become smooth functions of energy which drastically alters the behavior of the metric $g_{\tau\eta}$. To verify this hypothesis, we need first to note that the eigenvectors of $g_{\tau\eta}$, denote them by v^k, are dual to operators O_k, defined as

$$O_k = \sum_{\tau\in \mathsf{P}_N^2} v_\tau^k \tau. \qquad [9]$$

Numerical diagonalization of the metric indeed confirms that operators O_k have smooth expectation values in the energy eigenbasis of H'. For example, Fig. 4 depicts the behavior of the expectation values $\mathcal{E}_2 = \langle n|O_2|n\rangle$ of the operator O_2 corresponding to second eigenvector v^2, as a function of energy $\mathcal{E}$, showing that $\mathcal{E}_2(\mathcal{E})$ becomes a smooth function of $\mathcal{E}$ with increasing system size N.

The functional behavior of eigenoperator is also rather simple, which begs the question of whether we can understand the spectrum of $g_{\tau\eta}$ in more detail. First of all, it is easy to check that h^0 is an eigenvector of $g_{\tau\eta}$ corresponding to the largest eigenvalue $\lambda_1 = 1$:

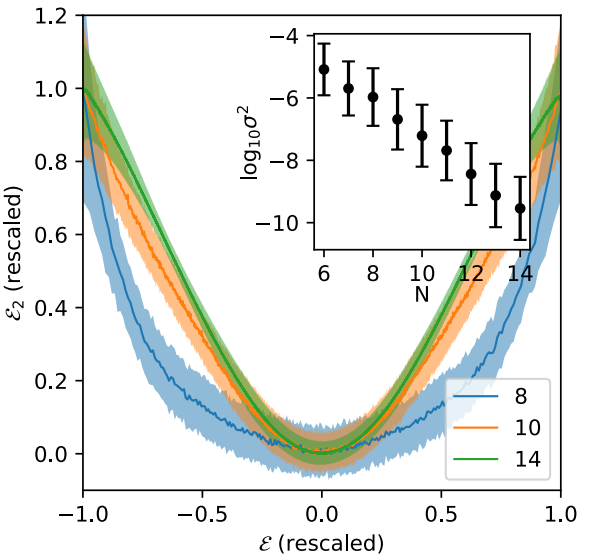

Fig. 4. Expectation value of the second eigenoperator of $g_{\tau\eta}$ defined by expression Eq. **9** in the eigenstates of the 2-local Hamiltonian H'. The figure shows $\langle n|O_2|n\rangle$ as a function of the eigenenergy $\mathcal{E}_n$. The line is the mean over the 200 realizations. The shaded region is the SD. The subplot shows the residual from a sixth-order polynomial fit to the data, which is used to subtract the smooth part of the result. One clearly observes an exponential suppression of the fluctuations with system size.

$$\begin{aligned}\sum_\eta g_{\tau\eta}h_\eta^0 &= \frac{1}{2^N}\sum_n \langle n|\,\tau\,|n\rangle\,\langle n|\sum_{\eta\in\mathsf{P}_N^2} h_\eta^0\eta\,|n\rangle \\ &= \frac{1}{2^N}\sum_n \langle n|\,\tau\,|n\rangle\,\langle n|\,H\,|n\rangle \\ &= \frac{1}{2^N}\sum_n E_n\,\langle n|\,\tau\,|n\rangle = h_\tau^0, \end{aligned} \qquad [10]$$

where, in the last step, we used the fact the gradient in Eq. **5** is zero when evaluated in h^0. This means that a perturbation in the direction of h^0 increases the cost function significantly. However, the associated operator O_1 is just the Hamiltonian H' itself. Such perturbations thus only result in a rescaling of the Hamiltonian. It is obvious why this increases the cost C. Nevertheless, since this can just be absorbed in a redefinition of time it leaves all the physics invariant. It is more interesting to understand the behavior of the subleading eigenvalues.

In general, the full set of eigenvalues of $g_{\tau\eta}$ is given by (*SI Appendix*, section 4)

$$\lambda_k = \frac{1}{2^N}\frac{\sum_{\tau\in\mathsf{P}_N^2}\mathrm{Tr}\,(F_k(H')\tau)^2}{\mathrm{Tr}\,(F_k(H')^2)}, \qquad [11]$$

where $F_k(H')$ is a function of 2-local H'. The eigenfunctions F_k need not be well behaved, like in the diagonal case described earlier. However, assuming the eigenstates of H' obey ETH, these functions should be smooth. We have already established that $F_1(x) = x$ and we know all operators need to be traceless. Furthermore, the eigenvalues, given by Eq. **11** have a simple interpretation. They are the square Frobenius norm of (normalized) projection of $F_k(H')$ on the two local subspace P_N^2. Powers of H' generate more and more nonlocal strings, which suggests that the eigenoperators are close to projected orthogonal polynomials of H'. We indeed find that the F_k can be very well approximated by Gram–Schmidt orthogonalization of polynomial function of H' of degree k, starting with $F_0(x) = \mathbb{1}$

(to ensure tracelessness) and $F_1(x) = x$. Thus, for example, $F_2(x)$ is the traceless part of x^2, i.e.,

$$F_2(x) = x^2 - \frac{\mathrm{Tr}(x^2)}{2^N}. \quad [12]$$

The results for the first few eigenvalues of $g_{\tau\eta}$ are shown in Fig. 5, together with the exact eigenvalues. The largest eigenvalue is indeed 1, and all the other eigenvalues decay rapidly with N for small systems. Some of the large eigenvalues appear to recover or slow down for larger systems. Note that we find excellent agreement between the approximate eigenvalues constructed from F_k and the exact results at larger N. In addition, these eigenvalues are variational so they form a lower bound to the true eigenvalues.

In principle, closed-form expression for the eigenvalues can be extracted exactly from expression Eq. **11** in terms of the coupling constants h; the problem is entirely algebraic. Nonetheless, the general problem is rather cumbersome to say the least. To make further progress, we restrict ourselves to compute λ_2 from Eqs. **11** and **12** under the assumption that the model is diagonal, i.e., H' is an Ising model with coupling J (we verify in *SI Appendix*, sections 5 and 6 and Fig. S2 that this approximation does not affect the qualitative behavior). A detailed diagrammatic calculation is presented in *SI Appendix*, section 5, and the result reads:

$$\lambda_2 = \frac{2\,\mathrm{Tr}(J^4) - \sum_i [(J^2)_{ii}]^2}{3\,\mathrm{Tr}J^4 + \frac{1}{2}\,\mathrm{Tr}(J^2)^2 - 6\sum_i [(J^2)_{ii}]^2 + 2\sum_{ij}(J_{ij})^4}, \quad [13]$$

which, for large N, becomes

$$\lambda_2 \sim \frac{2\,\mathrm{Tr}(J^4)}{3\,\mathrm{Tr}J^4 + \frac{1}{2}\,\mathrm{Tr}(J^2)^2} \leq 4\frac{\mathrm{Tr}(J^4)}{\mathrm{Tr}(J^2)^2}. \quad [14]$$

The latter can be interpreted as the inverse participation ratio or purity of the spectrum of the matrix J^2 (31). Thus, if all eigenvalues μ_J participate equally to λ_2 then $\lambda_2 = O(N^{-1})$; on the contrary, if only few of the μ_J's participate, then $\lambda_2 = O(1)$. This remark is of particular importance in that it may explain the emergence of 2-locality altogether. If $\lambda_2 \to 0$, then any perturbation of the coupling constants h is marginally irrelevant, meaning that it will not change the spectrum of the theory when $N \to \infty$. These coupling constants are thus by no means fined tuned to any specific value: They just happen to have a particular value, but the volume of allowed values they can take on, leaving the physics unchanged, is enormous. Alternatively, λ_2 tends to a constant in the thermodynamic limit, which means that J can be approximated by a low-rank matrix. The numerical data in Figs. 5 and 6 suggest this might be the case. Arguments can be given either way, on the one hand, it seems expected that one has to put a few more constraints, other than the bandwidth to stay close to the desired density of states. On the other hand, we have also explicitly computed the cost function for low-rank J in *SI Appendix*, section 6 (*SI Appendix*, Fig. S3), which suggests the cost function saturates at a constant at fixed rank, implying finite error for localization for large N. Regardless of the final outcome, our results lead to the remarkable observation that one can either 2-localize a GOE random matrix on a model with a finite number of parameters or there is large emergent invariance.

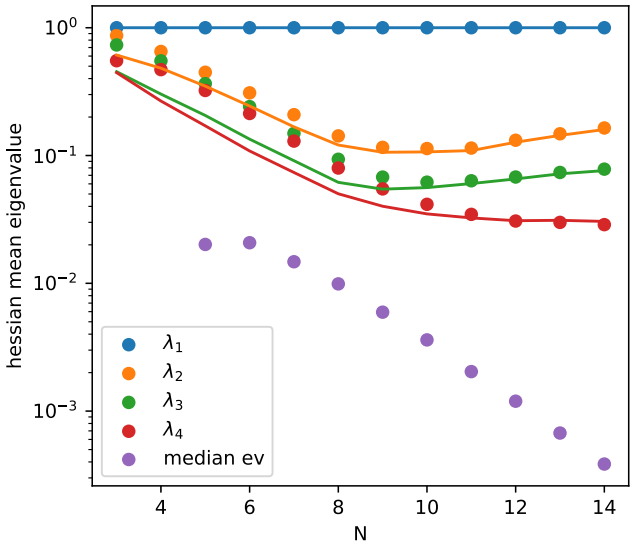

Fig. 5. The four largest eigenvalues λ_k of the hessian $g_{\tau\eta}$ as a function of system size N. The largest eigenvalue $\lambda_1 = 1$ corresponds to the eigenvector $v^1 \equiv h^0_\tau$ (Eq. **10**). The subsequent eigenvalues are all smaller than one. Their large N behavior is especially important to understand how fine-tuned the coupling constants $J^{\alpha\beta}_{ij}$ of the 2-local Hamiltonian H' are (i.e., the "sloppiness" of the 2-local Universe); hence their asymptotic behavior is discussed extensively in the main text, where we suggest a plausible scenario and its implications. Each dot corresponds to an average over 200 realizations. Solid lines are estimates using expression Eq. **11** where $F_k(H)$ is constructed using the procedure explained in the main text.

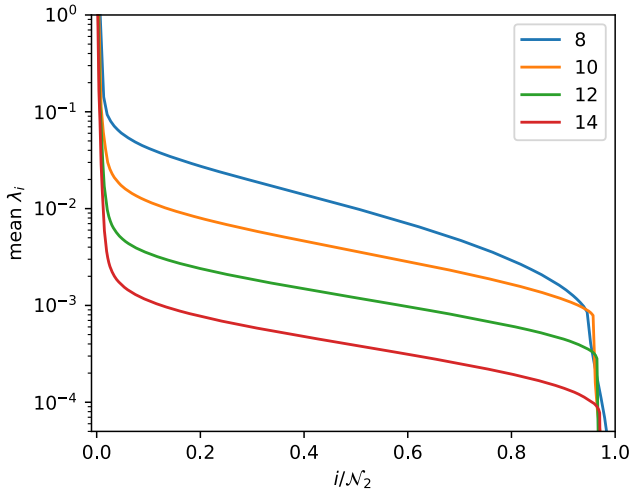

Fig. 6. Distribution of the ordered (decreasing order) eigenvalues λ_i of the Hessian $g_{\tau\eta}$ for different system sizes N. For clarity, the results are shown as a function of $i/\mathcal{N}_2$, where $\mathcal{N}_2$ is the number of 2-local Pauli strings, i.e the dimension of the Hessian. Only a few large eigenvalues contribute, as shown in detail in Fig. 5. The typical eigenvalue decreases exponentially decreases with N, making the model very sloppy.

Before we conclude, let us stress that, even when λ_2 is nonzero, 2-local Hamiltonians are very sloppy (32). That is, most combinations of parameters h are unimportant, as shown in Fig. 6. This conclusion is further supported by a simple generalization of expression Eq. **14** to higher k. While it is rather cumbersome to establish the full result, the leading order contribution behaves like $\lambda_k \sim \mathrm{Tr}(J^{2k})/\mathrm{Tr}(J^2)^k$. As a result, the eigenvalues of the metric are expected to follow a geometric progression, the hallmark of "sloppiness" (32).

B. Geometric locality. The above results are only concerned with few-body nature of the Hamiltonian, i.e., an arbitrary 2-local Hamiltonian has all-to-all interactions. However, the sloppiness of the resulting ensemble suggests the model can still be greatly simplified without fine-tuning. In particular, Fig. 6 suggests there are only $O(N)$ relevant parameters. In supplement, we sparsify the coupling by adding an l_1 penalty to our cost function, a common practice in spare regression. It is shown that we can

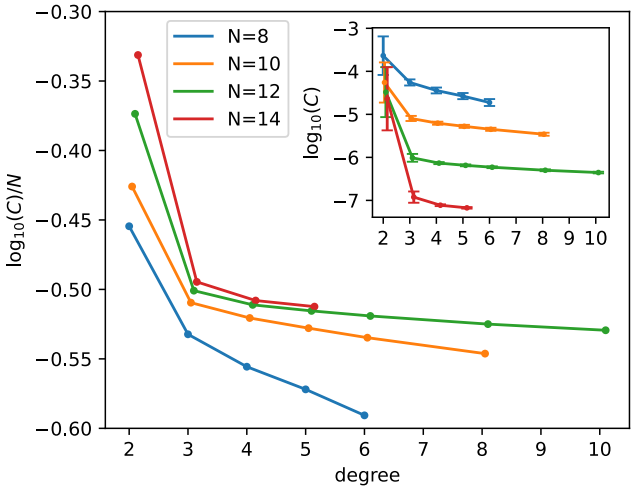

Fig. 7. Localization error C (as defined by expression Eq. **4**) versus graph degree after localization of a GOE matrix on a random regular graph. Each datapoint is an average of 200 realizations, realizations meaning different GOE matrices and different graphs. Remarkably, little loss is observed as compared to the complete graph by restricting the interactions to a finite low degree ($d \sim 4$) interaction graphs.

achieve high sparsity at almost no cost. To further substantiate the findings, and make them more tangible, we 2-localize matrices on a random regular graph (RRG) of degree d. For geometric locality to emerge, 2-localization should be robust to constraining to connectivity of the interaction graph. In Fig. 7, we show results for localization of GOE spectra (see *SI Appendix* for details on the method). We find that low connectivity ($\sim$4) regular graphs are enough to localize GOE spectra with a precision that only differs from the complete graph by subleading corrections in the system size N.

4. Conclusion and Discussion

We find that random GOE matrices can be represented in a local form with very good precision, i.e., the norm of the remaining nonlocal part decreases exponentially with the size of the system. This effectively corresponds to an exponential compression of the amount of data. Among other things, this is a step toward the resolution of the preferred basis problem: Associated with each random Hamiltonian, there is a preferred basis in which this Hamiltonian has an almost local description.

Generic 2-local Hamiltonian has no geometric structure, but because the problem is sloppy, the additional constraints set by geometric locality can be incorporated without affecting the quality of the results. For example, low connectivity random regular graphs are already an excellent model for GOE spectra.

This suggests a route to understand how space could emerge from quantum mechanics alone by looking at the adjacency structure of the couplings J_{ij} (28, 29, 33–35). On the other hand, we also showed that even if generic random Hamiltonians can be localized efficiently, some particular Hamiltonians cannot. These are examples of operators that have some fundamental quantum nonlocal properties and cannot be represented as a sum of 2-body operators.

We note that our theoretical framework is not limited in its scope to random matrix models from the GOE (or GUE) ensembles, but can be successfully applied also to explicit multispin Hamiltonians such as the SYK model, in that their spectrum can be accurately reproduced by a two-localized model with exponential precision. A full discussion of the 2-local analog of the SYK model will be published elsewhere.

Data, Materials, and Software Availability. A minimal 2-localization python code aside with our main numerical results is available at https://github.com/nicolasloizeau/Quantum-Localization (36). All other data are included in the manuscript and/or *SI Appendix*.

ACKNOWLEDGMENTS. The Flatiron Institute is a division of the Simons Foundation. We acknowledge support from Air Force Office of Scientific Research (AFOSR): Grant FA9550-21-1-0236. This work was supported in part through the New York University (NYU) IT High Performance Computing resources, services, and staff expertise.

1. P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford, ed. 4, 1930).
2. D. Sels, M. Wouters, Quantum equivalence, the second law and emergent gravity. arXiv [Preprint] (2014). https://arxiv.org/abs/1411.3901 (Accessed 3 September 2023).
3. S. M. Carroll, *Reality as a Vector in Hilbert Space*, V. Allori, Ed. (Springer International Publishing, Cham, 2022), pp. 211–224.
4. S. M. Carroll, A. Singh, *Mad-Dog Everettianism: Quantum Mechanics at Its Most Minimal*, A. Aguirre, B. Foster, Z. Merali, Eds. (Springer International Publishing, Cham, 2019), pp. 95–104.
5. S. Weinberg, *The Quantum Theory of Fields: Foundations* (Cambridge University Press, 2005), vol. 1.
6. P. Zanardi, D. A. Lidar, S. Lloyd, Quantum tensor product structures are observable induced. *Phys. Rev. Lett.* **92**, 060402 (2004).
7. E. Schrödinger, Discussion of probability relations between separated systems. *Math. Proc. Cambridge Philos. Soc.* **31**, 555–563 (1935).
8. E. Joos, H. D. Zeh, The emergence of classical properties through interaction with the environment. *Zeit. Phys. B Condens. Matter* **59**, 223–243 (1985).
9. W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys.* **75**, 715–775 (2003).
10. M. Schlosshauer, Decoherence, the measurement problem, and interpretations of quantum mechanics. *Rev. Mod. Phys.* **76**, 1267–1305 (2005).
11. J. S. Cotler, G. R. Penington, D. H. Ranard, Locality from the spectrum. *Commun. Math. Phys.* **368**, 1267–1296 (2019).
12. M. V. Berry, "A half-century of physical asymptotics and other diversions" in *Quantum Chaology (The Bakerian Lecture)* (World Scientific, 2017), pp. 307–322.
13. F. Haake, *Quantum Signatures of Chaos* (Springer, Heidelberg, 2001).
14. M. Freedman, M. S. Zini, The universe from a single particle. *J. High Energy Phys.* **2021**, 140 (2021).
15. M. Freedman, M. S. Zini, The universe from a single particle. Part II. *J. High Energy Phys.* **2021**, 102 (2021).
16. E. P. Wigner, Characteristic vectors of bordered matrices with infinite dimensions. *Ann. Math.* **62**, 548–564 (1955).
17. T. Guhr, A. Müller-Groeling, H. A. Weidenmüller, Random-matrix theories in quantum physics: Common concepts. *Phys. Rep.* **299**, 189–425 (1998).
18. Y. Y. Atas, E. Bogomolny, O. Giraud, G. Roux, Distribution of the ratio of consecutive level spacings in random matrix ensembles. *Phys. Rev. Lett.* **110**, 84101 (2013).
19. C. G. Broyden, The convergence of a class of double-rank minimization algorithms 1. General considerations. *IMA J. Appl. Math.* **6**, 76–90 (1970).
20. R. Fletcher, A new approach to variable metric algorithms. *Comput. J.* **13**, 317–322 (1970).
21. D. Goldfarb, A family of variable-metric methods derived by variational means. *Math. Comput.* **24**, 23–26 (1970).
22. D. F. Shanno, Conditioning of quasi-Newton methods for function minimization. *Math. Comput.* **24**, 647–656 (1970).
23. R. Fletcher, *Practical Methods of Optimization* (John Wiley Sons, Ltd, 2000).
24. I. Dumitriu, A. Edelman, Matrix models for beta ensembles. *J. Math. Phys.* **43**, 5830–5847 (2002).
25. M. L. Mehta, *Random Matrices, Pure and Applied Mathematics* (Elsevier, 2004), vol. 142, pp. xiii–xiv.
26. J. Šuntajs, J. Bonča, T. Prosen, L. Vidmar, Quantum chaos challenges many-body localization. *Phys. Rev. E* **102**, 062144 (2020).
27. A. Prakash, J. H. Pixley, M. Kulkarni, Universal spectral form factor for many-body localization. *Phys. Rev. Res.* **3**, L012019 (2021).
28. J. S. Cotler *et al.*, Black holes and random matrices. *J. High Energy Phys.* **2017**, 118 (2017).
29. J. Maldacena, D. Stanford, Remarks on the Sachdev-Ye-Kitaev model. *Phys. Rev. D* **94**, 106002 (2016).
30. L. D'Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Adv. Phys.* **65**, 239–362 (2016).
31. R. J. Bell, P. Dean, Atomic vibrations in vitreous silica. *Discuss. Faraday Soc.* **50**, 55–61 (1970).
32. B. B. Machta, R. Chachra, M. K. Transtrum, J. P. Sethna, Parameter space compression underlies emergent theories and predictive models. *Science* **342**, 604–607 (2013).
33. C. Cao, S. M. Carroll, S. Michalakis, Space from Hilbert space: Recovering geometry from bulk entanglement. *Phys. Rev. D* **95**, 24031 (2017).
34. S. M. Carroll, A. Singh, Quantum mereology: Factorizing Hilbert space into subsystems with quasiclassical dynamics. *Phys. Rev. A* **103**, 022213 (2021).
35. J. D. Bekenstein, Black holes and entropy. *Phys. Rev. D* **7**, 2333–2346 (1973).
36. N. Loizeau, Quantum localization. Github. https://github.com/nicolasloizeau/Quantum-Localization. Deposited 11 August 2023.