As one expects in this case, the better the scattering environment can distinguish between system states $|\alpha\rangle$ and $|\beta\rangle$ the more coherence is lost in this elastic process.

In the general case, the decay of off-diagonal elements will be due to a combination of elastic and inelastic processes. Although little can be said without specifying the interaction, it is clear that the integral over $|f_{\alpha\alpha} - f_{\beta\beta}|^2$ in (5.184), a "decoherence cross section" without classical interpretation, is not related to the inelastic cross sections characterizing the population transfer, and may be much larger. In this case, the resulting decoherence will be again much faster than the corresponding relaxation time scales.

5.5 Robust States and the Pointer Basis

We have seen that, even though the decoherence predictions of linear coupling models has to be taken with great care, the general observation remains valid that the loss of coherence may occur on a time scale $\gamma_{\text{deco}}^{-1}$ that is much shorter the relaxation time γ^{-1}. Let us therefore return to the general description of open systems in terms of a semigroup generator $\mathcal{L}$, and ask what we can say about a general state after a time t which is still small compared to the relaxation time, but much larger than the decoherence time scale. From a classical point of view, which knows only about relaxation, the state has barely changed, but in the quantum description it may now be well approximated by a mixture determined by particular projectors P_ℓ,

$$\mathrm{e}^{\mathcal{L}t} : \rho \xrightarrow{\gamma_{\text{deco}}^{-1} \ll t \ll \gamma^{-1}} \rho_t \simeq \rho' = \sum_\ell \operatorname{tr}(\rho \mathsf{P}_\ell)\, \mathsf{P}_\ell. \tag{5.185}$$

This set of projectors $\{\mathsf{P}_\ell\}$, which depend at most weakly on t, is called *pointer basis* [43] or set of *robust states* [44]. It is distinguished by the fact that a system prepared in such a state is hardly affected by the environment, while a superposition of two distinct pointer states decoheres so rapidly that it is never observed in practice.

We encountered this behavior with the damped harmonic oscillator discussed in Sect. 5.3.4. There the coherent oscillator states remained pure under Markovian dynamics, while superpositions between (macroscopically distinct) coherent states decayed rapidly. Hence, in this case the coherent states $\mathsf{P}_\alpha = |\alpha\rangle\langle\alpha|$ can be said to form an (over-complete) set of robust states, leading to the mixture

$$\rho' = \int \mathrm{d}\mu\,(\alpha) \operatorname{tr}(\rho \mathsf{P}_\alpha) \mathsf{P}_\alpha\,, \tag{5.186}$$

with appropriate measure μ.

The name *pointer basis* is well-fitting because the existence of such robust states is a prerequisite for the description of an ideal measurement device in a quantum framework. A macroscopic – and therefore decohering – apparatus implementing the measurement of an observable A is ideally constructed in such a way that macroscopically distinct positions of the "pointer" are obtained for the different eigenstates of A. Provided these pointer positions of the device are robust, the correct values are observed with certainty if the quantum system is in an eigenstate of the observable. Conversely, if the quantum system is not in an eigenstate of A, the apparatus will *not* end up in a superposition of pointer positions, but be found at a definite position, albeit probabilistically, with a probability given by the Born rule.

The main question regarding pointer states is, given the environmental coupling or the generator $\mathcal{L}$, what determines whether a state is robust or not, and how can we determine the set of pointer states without solving the master equation for all initial states. It is fair to say that this issue is not fully understood, except for very simple model environments, nor is it even clear how to quantify robustness.

An obvious ansatz, due to Zurek [6, 45], is to sort all pure states in the Hilbert space according to their (linear) entropy production rate, or rate of loss of purity,

$$\partial_t \mathsf{S}_{\mathrm{lin}}[\rho] = -2\,\mathrm{tr}\,(\rho\mathcal{L}(\rho)) \ . \tag{5.187}$$

It has been called "predictability sieve" since the least entropy producing and therefore most predictable states are candidate pointer states [6].

In the following, a related approach will be described, following the presentation in [3, 46]. It is based on a time-evolution equation for robust states. Since such an equation must distinguish particular states from their linear superpositions it is necessarily nonlinear.

5.5.1 Nonlinear Equation for Robust States

We seek a nonlinear time-evolution equation for robust pure states P_t which, on the one hand, preserves their purity, and on the other, keeps them as close as possible to the evolved state following the master equation.

A simple nonlinear equation keeping a pure state pure is given by the following extension of the Heisenberg form for the infinitesimal time step,

$$\mathsf{P}_{t+\delta t} = \mathsf{P}_t + \delta t\left(\frac{1}{\mathrm{i}}[\mathsf{A}_t, \mathsf{P}_t] + [\mathsf{P}_t, [\mathsf{P}_t, \mathsf{B}_t]]\right) , \tag{5.188}$$

where A and B are hermitian operators. In fact, the unitary part can be absorbed into the nonlinear part by introducing the hermitian operator $\mathsf{X}_t = -\mathrm{i}[\mathsf{A}_t, \mathsf{P}_t] + \mathsf{B}_t$. It "generates" the infinitesimal time translation of the projectors (and may be a function of P_t),

$$\mathsf{P}_{t+\delta t} = \mathsf{P}_t + \delta t[\mathsf{P}_t, [\mathsf{P}_t, \mathsf{X}_t]] \ . \tag{5.189}$$

With this choice one confirms easily that the evolved operator has indeed the properties of a projector, to leading order in δt,

$$\mathsf{P}_{t+\delta t}^{\dagger} = \mathsf{P}_{t+\delta t} \tag{5.190}$$

and

$$\left(\mathsf{P}_{t+\delta t}\right)^2 = \mathsf{P}_{t+\delta t} + O(\delta t^2)\,. \tag{5.191}$$

The corresponding differential equation reads

$$\partial_t \mathsf{P}_t = \frac{\mathsf{P}_{t+\delta t} - \mathsf{P}_t}{\delta t} = \left[\mathsf{P}_t, [\mathsf{P}_t, \mathsf{X}_t]\right]\,. \tag{5.192}$$

To determine the operator X_t one minimizes the distance between the time derivatives of the truly evolved state and the projector. If we visualize the pure states as lying on the boundary of the convex set of mixed states, then a pure state will in general dive into the interior under the time evolution generated by $\mathcal{L}$. The minimization chooses the operator X_t in such a way that P_t sticks to the boundary, while remaining as close as possible to the truly evolved state.

The (Hilbert–Schmidt) distance between the time derivatives can be calculated as

$$\begin{aligned}\| \underbrace{\mathcal{L}(\mathsf{P}_t)}_{\equiv \mathsf{Z}} - \partial_t \mathsf{P}_t \|_{\mathrm{HS}}^2 &= \operatorname{tr}\left[\left(\mathsf{Z} - [\mathsf{P}_t, [\mathsf{P}_t, \mathsf{X}_t]]\right)^2\right] \\ &= \operatorname{tr}\left(\mathsf{Z}^2 - 2(\mathsf{Z}^2\mathsf{P}_t - (\mathsf{Z}\mathsf{P}_t)^2)\right) \\ &\quad + 2\operatorname{tr}\left((\mathsf{Z}-\mathsf{X})^2\mathsf{P}_t - ((\mathsf{Z}-\mathsf{X})\,\mathsf{P}_t)^2\right)\,. \end{aligned} \tag{5.193}$$

We note that the first term is independent of X, whereas the second one is non-negative. With the obvious solution $\mathsf{X}_t = \mathsf{Z} \equiv \mathcal{L}(\mathsf{P}_t)$ one gets a nonlinear evolution equation for robust states P_t, which is trace and purity preserving [46],

$$\partial_t \mathsf{P}_t = \left[\mathsf{P}_t, [\mathsf{P}_t, \mathcal{L}(\mathsf{P}_t)]\right]. \tag{5.194}$$

It is useful to write down the equation in terms of the vectors $|\xi\rangle$ which correspond to the pure state $\mathsf{P}_t = |\xi\rangle\langle\xi|$,

$$\partial_t |\xi\rangle = [\mathcal{L}(|\xi\rangle\langle\xi|) - \underbrace{\langle\xi|\mathcal{L}(|\xi\rangle\langle\xi|)|\xi\rangle}_{\text{"decay rate"}}]|\xi\rangle\,. \tag{5.195}$$

If we take $\mathcal{L}$ to be of the Lindblad form (5.79) the equation reads

$$\begin{aligned}\partial_t|\xi\rangle = {}& \frac{1}{\mathrm{i}\hbar}\mathsf{H}|\xi\rangle + \sum_k \gamma_k \left[\langle \mathsf{L}_k^\dagger\rangle_\xi \left(\mathsf{L}_k - \langle \mathsf{L}_k\rangle_\xi\right) - \frac{1}{2}\left(\mathsf{L}_k^\dagger \mathsf{L} - \langle \mathsf{L}_k^\dagger \mathsf{L}_k\rangle\right)\right]|\xi\rangle \\ & - \frac{1}{\mathrm{i}\hbar}\langle \mathsf{H}\rangle_\xi |\xi\rangle\,. \end{aligned} \tag{5.196}$$

Its last term is usually disregarded because it gives rise only to an additional phase if $\langle \mathsf{H}\rangle_\xi$ is constant. The meaning of the nonlinear equation (5.196) is best studied in terms of concrete examples.

5.5.2 Applications

Damped Harmonic Oscillator

Let us start with the damped harmonic oscillator discussed in Sect. 5.3.4. By setting $\mathsf{H} = \hbar\omega \mathsf{a}^\dagger \mathsf{a}$ and $\mathsf{L} = \mathsf{a}$ (5.196) turns into

$$\partial_t |\xi\rangle = -\mathrm{i}\omega \mathsf{a}^\dagger \mathsf{a} |\xi\rangle + \gamma \left(\langle \mathsf{a}^\dagger \rangle_\xi (\mathsf{a} - \langle \mathsf{a} \rangle_\xi) - \frac{1}{2} \left(\mathsf{a}^\dagger \mathsf{a} - \langle \mathsf{a}^\dagger \mathsf{a} \rangle_\xi \right) \right) |\xi\rangle \,. \tag{5.197}$$

Note that the first term of the non-unitary part vanishes if $|\xi\rangle$ is a coherent state, i.e., an eigenstate of a. This suggests the ansatz $|\xi\rangle = |\alpha\rangle$ which leads to

$$\partial_t |\alpha\rangle = \left[\left(-\mathrm{i}\omega - \frac{\gamma}{2} \right) \alpha \mathsf{a}^\dagger + \frac{\gamma}{2} |\alpha|^2 \right] |\alpha\rangle \,. \tag{5.198}$$

It is easy to convince oneself that this equation is solved by

$$|\alpha_t\rangle = |\alpha_0 \mathrm{e}^{-\mathrm{i}\omega t - \gamma t/2}\rangle = \mathrm{e}^{-|\alpha_t|^2/2} \mathrm{e}^{\alpha_t \mathsf{a}^\dagger} |0\rangle \tag{5.199}$$

with $\alpha_t = \alpha_0 \exp(-\mathrm{i}\omega t - \gamma t/2)$. It shows that the predicted robust states are indeed given by the slowly decaying coherent states encountered in Sect. 5.3.4.

Quantum Brownian Motion

A second example is given by the Brownian motion of a quantum particle. The choice

$$\mathsf{H} = \frac{\mathsf{p}^2}{2m} \quad \text{and} \quad \mathsf{L} = \frac{\sqrt{8\pi}}{\Lambda_{\mathrm{th}}} \mathsf{x} \tag{5.200}$$

yields a master equation of the form (5.117) but without the dissipation term. Inserting these operators into (5.196) leads to

$$\partial_t |\xi\rangle = \frac{\mathsf{p}^2}{2m\mathrm{i}\hbar} |\xi\rangle - \gamma \frac{4\pi}{\Lambda_{\mathrm{th}}^2} [(\mathsf{x} - \langle \mathsf{x} \rangle_\xi)^2 - \underbrace{\langle (\mathsf{x} - \langle \mathsf{x} \rangle_\xi)^2 \rangle_\xi}_{\sigma_\xi^2(\mathsf{x})}] |\xi\rangle \,. \tag{5.201}$$

The action of the non-unitary term is apparent in the position representation, $\xi(x) = \langle x|\xi\rangle$. At positions x which are distant from mean position $\langle \mathsf{x} \rangle_\xi$ as compared to the dispersion $\sigma_\xi(\mathsf{x}) = \left\langle (\mathsf{x} - \langle \mathsf{x} \rangle_\xi)^2 \right\rangle_\xi^{1/2}$ the term is negative and the value $\xi(x)$ gets suppressed. Conversely, the part of the wave function close to the mean position gets enhanced,

$$\langle x|\xi\rangle = \begin{cases} \text{suppressed if } |x - \langle \mathsf{x} \rangle_\xi| > \sigma_\xi(\mathsf{x}) \\ \text{enhanced} \quad \text{if } |x - \langle \mathsf{x} \rangle_\xi| < \sigma_\xi(\mathsf{x}) \,. \end{cases} \tag{5.202}$$

This localizing effect is countered by the first term in (5.201) which causes the dispersive broadening of the wave function. Since both effects compete we expect stationary, soliton-like solutions of the equation.

Indeed, a Gaussian ansatz for $|\xi\rangle$ with ballistic motion, i.e., $\langle \mathsf{p}\rangle_\xi = p_0$, $\langle \mathsf{x}\rangle_\xi = x_0 + p_0 t/m$, and a *fixed* width $\sigma_\xi(\mathsf{x}) = \sigma_0$ solves (5.201) provided [44]

$$\sigma_0^2 = \frac{1}{4\pi}\sqrt{\frac{k_B T}{2\hbar\gamma}}\Lambda_{\mathrm{th}}^2 = \left(\frac{\hbar^3}{8\gamma m^2 k_B T}\right)^{1/2}, \tag{5.203}$$

see (5.128). As an example, let us consider a dust particle with a mass of 10 µg in the interstellar medium interacting only with the microwave background of $T = 2.7\,\mathrm{K}$. Even if we take a very small relaxation rate of $\gamma = 1/(13.7\times 10^9\,\mathrm{y})$, corresponding to the inverse age of the universe, the width of the solitonic wave packet describing the center of mass is as small as 2 pm. This sub-atomic value demonstrates again the remarkable efficiency of the decoherence mechanism to induce classical behavior in the quantum state of macroscopic objects.

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