Quantized vector potential and alternative views of the magnetic Aharonov-Bohm phase shift

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We give a complete quantum analysis of the Aharonov-Bohm (AB) magnetic phase shift involving three entities: the electron, the charges constituting the solenoid current, and the vector potential. The usual calculation supposes that the solenoid's vector potential may be well approximated as classical. The AB shift is then acquired by the quantized electron moving in this vector potential. Recently, Vaidman presented a semiclassical calculation [L. Vaidman, Phys. Rev. A **86**, 040101 (2012)], later confirmed by a fully quantum calculation of Pearle and Rizzi [preceding paper, *ibid.* **95**, 052123 (2017)], where it is supposed that the electron's vector potential may be well approximated as classical. The AB shift is then acquired by the quantized solenoid charges moving in this vector potential. Here we present a third calculation, which supposes that the electron and solenoid currents may be well approximated as classical sources. The AB phase shift is then shown to be acquired by the quantized vector potential. We next show these are three equivalent alternative ways of calculating the AB shift. We consider the exact problem where all three entities are quantized. We approximate the wave function as the product of three wave functions: a vector potential wave function, an electron wave function, and a solenoid wave function. We apply the variational principle for the exact Schrödinger equation to this approximate form of solution. This leads to three Schrödinger equations, one each for vector potential, electron, and solenoid, each with classical sources for the other two entities. However, each Schrödinger equation contains an additional real *c*-number term, the time derivative of an extra phase. We show that these extra phases are such that the phase of the total wave function produces the AB shift. Since none of the three entities requires different treatment from any of the others, this leads to three alternative views of the physical cause of the AB magnetic effect.

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I. INTRODUCTION

The exact problem for the magnetic Aharonov-Bohm (AB) effect involves three quantized entities: the electron, the solenoid charges, and the vector potential. The Schrödinger equation has the form

$$i\frac{d}{dt}|\psi,t\rangle = \bigg[\hat{H}_{\rm el} + \hat{H}_{\rm sol} + \hat{H}_A - \int d\mathbf{x}[\hat{\mathbf{J}}_{\rm el}(\mathbf{x}) + \hat{\mathbf{J}}_{\rm sol}(\mathbf{x})]\cdot\hat{\mathbf{A}}(\mathbf{x})\bigg]|\psi,t\rangle, \quad (1)$$

where $\hat{H}_{\rm el}$ and $\hat{H}_{\rm sol}$ are Hamiltonians for the electron and solenoid charges. In addition to the kinetic energy terms, if the particles move in tubes or start at rest and are accelerated to some final speed, this will include potentials was well. Here $\hat{H}_A$ is the Hamiltonian for the free quantum vector potential field. Further, $\hat{\mathbf{J}}_{\rm el}(\mathbf{x})$ and $\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})$ are the current operators of the electron and solenoid charges, respectively, and $\hat{\mathbf{A}}(\mathbf{x})$ is the vector potential operator.

One would expect that the phase shift calculated thereby would be the usual AB expression if, for no other reason, that experiment shows this result [1]. However, this exact problem has not been solved.

What can be solved are three truncated problems, detailed in the following three paragraphs. In each of these problems, in the interaction term, two of these entities are considered classical and one is considered quantum.

In the standard treatment, based upon the original work by Aharonov and Bohm [2–4], the choice is made to approximate the solenoid current as classical, $\hat{\mathbf{J}}_{\rm sol}(\mathbf{x}) \to \mathbf{J}_{\rm sol}(\mathbf{x})$, where $\mathbf{J}_{\rm sol}(\mathbf{x})$ is the expectation value of $\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})$. The vector potential due to the solenoid is also approximated as classical, $\hat{\mathbf{A}}(\mathbf{x}) \to \mathbf{A}_{\rm sol}(\mathbf{x})$, with $\mathbf{J}_{\rm sol}(\mathbf{x})$ as source of $\mathbf{A}_{\rm sol}(\mathbf{x})$. Omitting the completely classical interaction term $\int d\mathbf{x}\,\mathbf{J}_{\rm sol}(\mathbf{x})\cdot\mathbf{A}_{\rm sol}(\mathbf{x})$, the resulting interaction term is therefore $\int d\mathbf{x}\,\hat{\mathbf{J}}_{\rm el}(\mathbf{x})\cdot\mathbf{A}_{\rm sol}(\mathbf{x})$. Thus, only the quantized electron undergoes an interaction. Then the phase shift associated with the electron moving in this classical vector potential, the AB phase shift, was found.

One might cavil that such a vector potential can hardy be considered classical since, unlike anything classical, it is force-free yet has a physical effect. However, classicizing the vector potential is accepted and its effect on the quantized electron is taken to illustrate one of the marvelous distinctions between classical and quantum physics.

However, recently Vaidman [5] chose to approximate the electron current as classical, $\hat{\mathbf{J}}_{\rm el}(\mathbf{x}) \to \mathbf{J}_{\rm el}(\mathbf{x},t)$, where $\mathbf{J}_{\rm el}(\mathbf{x},t)$ is the expectation value of $\hat{\mathbf{J}}_{\rm el}(\mathbf{x})$. Moreover, the vector potential due to the electron is also approximated as classical, $\hat{\mathbf{A}}(\mathbf{x}) \to \mathbf{A}_{\rm el}(\mathbf{x},t)$, with $\mathbf{J}_{\rm el}(\mathbf{x},t)$ as source of $\mathbf{A}_{\rm el}(\mathbf{x},t)$. Omitting the completely classical interaction term $\int d\mathbf{x}\,\mathbf{J}_{\rm el}(\mathbf{x},t)\cdot\mathbf{A}_{\rm el}(\mathbf{x},t)$, the resulting interaction term is therefore $\int d\mathbf{x}\,\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})\cdot\mathbf{A}_{\rm el}(\mathbf{x},t)$. Thus, only the quantized solenoid charges undergo an interaction. Vaidman showed by a semiclassical calculation (verified by a fully-quantum-mechanical calculation [6]) that the solenoid charges provide a phase shift identical to the usual AB phase shift.

In Sec. II we choose to approximate both the solenoid current and the electron current as classical. The resulting

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interaction is therefore $\int d\mathbf{x}[\mathbf{J}_{\rm el}(\mathbf{x},t)+\mathbf{J}_{\rm sol}(\mathbf{x})]\cdot\hat{\mathbf{A}}(\mathbf{x})$. Thus, only the quantized vector potential undergoes an interaction. We then solve for the vector potential wave function. In Sec. III we show that the latter provides a phase shift identical to the usual AB phase shift.

We propose viewing these as three alternative but equally valid (although conceptually and mathematically different) ways of obtaining the same result. However, there is the following to be considered, which seems to raise an objection to this point of view.

There are three additional problems one might think of, in each of which only one entity is made classical in the interactions and the remaining two are treated as quantum mechanical. From the successes where two entities are made classical, one might expect to obtain the AB phase shift when only one entity is made classical.

Two of these problems are not readily soluble, so one cannot ascertain the phase shift for them, one where the electron and vector potential are quantized, the other where the solenoid charges and vector potential are quantized. However, the problem can be readily solved where the vector potential is treated classically and the evolution of the quantized electron and solenoid charges is governed by the Schrödinger equation

$$i\frac{d}{dt}|\psi,t\rangle=\bigg[\hat{H}_{\rm el}+\hat{H}_{\rm sol}-\int d\mathbf{x}[\hat{\mathbf{J}}_{\rm el}(\mathbf{x})\cdot\mathbf{A}_{\rm sol}(\mathbf{x})+\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})\cdot\mathbf{A}_{\rm el}(\mathbf{x},t)]\bigg]|\psi,t\rangle \quad (2)$$

[omitting the self-interacting terms $\int d\mathbf{x}\,\hat{\mathbf{J}}_{\rm el}(\mathbf{x})\cdot\mathbf{A}_{\rm el}(\mathbf{x})$ and $\int d\mathbf{x}\,\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})\cdot\mathbf{A}_{\rm sol}(\mathbf{x})$, which are the same for both traverses of the electron around the solenoid, left and right, and therefore do not contribute to the phase shift]. According to this equation, since the Hamiltonian is separable, both mechanisms are operating, the electron acquires the usual AB phase shift moving in the solenoid's vector potential, and the solenoid charges acquire the AB phase shift moving in the electron's vector potential. Thus, the net phase shift is twice the AB shift. So we see that the prescription of just letting the vector potential be classical is incorrect, at least in this case.

That the Schrödinger equation (2) is not the correct one to use for the problem of a jointly quantized electron and solenoid (with classical vector potential) was shown in Ref. [6]. There, a better approximation was found, starting with the variational principle for the Schrödinger equation, and the AB phase resulted (more details below).

Here, in Sec. IV, we consider the same prescription, a better approximation to the exact problem of a jointly quantized electron and solenoid and vector potential. We start with the variational principle for the Schrödinger equation, with the exact state vector replaced by the product $|\Psi,t\rangle\approx|\psi_A,t\rangle|\psi_{\rm el},t\rangle|\psi_{\rm sol},t\rangle$, where the operator dependence of $|\psi_A,t\rangle$ is just the vector potential, $|\psi_{\rm el},t\rangle$'s dependence is just the electron operators, and $|\psi_{\rm sol},t\rangle$'s dependence is just the solenoid operators. The variational principle produces three Schrödinger equations for the three state vectors. Each equation describes one entity interacting with classical (expectation) values of the other two entities, as well as an additional phase term.

There is the freedom to add phases to two state vectors and the negative of these two phases to the other state vector without affecting the overall phase of the wave function. We choose to use this to remove the extra phases from the electron and solenoid Schrödinger equations. Then, *except* for the extra phase term, $|\psi_A,t\rangle$ becomes the solution of the Schrödinger equation already discussed in Secs. II and III, whose phase contribution to the AB effect is, for the right traverse of the electron, correctly, $1/2$ the AB phase, $\equiv\Phi_R(T)$. The extra phase term is shown to provide $-2\Phi_R(T)$, so the net contribution to the phase of the vector potential Schrödinger equation *with* the extra phase is $-\Phi_R(T)$. Since both the electron and solenoid Schrödinger equations each produce $\Phi_R(T)$, the net phase of the product wave function of all three state vectors is, correctly, $\Phi_R(T)$.

It should be remarked that Ref. [6] considers the exact problem of the quantized electron and solenoid where the vector potential is not a quantum field but is a function of the electron and solenoid position and momentum operators. The same technique used here was used there, applying the variational principal, which gives the exact Schrödinger equation to an approximate state vector that is the direct product of electron and solenoid state vectors. Again, the two resulting Schrödinger equations contained extra phase terms. The net extra phase was shown to produce the negative AB shift. The wave functions for the electron and solenoid, without the extra phase, each give the AB shift and so the net result is that the AB shift is obtained. Therefore, these two papers show how a good approximative treatment of the exact Schrödinger equation describing the interaction between quantized entities responsible for the AB shift, either the electron and solenoid or the vector potential and electron and solenoid, results in the AB shift. Section V discusses some conclusions that may be drawn from these calculations.

We use natural units, with $c=\hbar=1$.

II. QUANTIZED VECTOR POTENTIAL WITH A CLASSICAL SOURCE

In this section the Hamiltonian to be considered is

$$\hat{H}=\int d\mathbf{x}\bigg[\frac{1}{2}\big[\hat{\pi}_i^2(\mathbf{x})+\nabla\hat{A}^i(\mathbf{x})\cdot\nabla\hat{A}^i(\mathbf{x})\big]-J^i(\mathbf{x},t)\hat{A}^i(\mathbf{x})\bigg], \quad (3)$$

where we are using the summation convention for repeated indices ($i=1,2,3$), $J^i(\mathbf{x},\mathbf{t})$ is a general classical current source (to be specialized in Sec. III, where the result is applied to the AB situation, to the sum of the electron current and the solenoid current), and $[\hat{A}^i(\mathbf{x}),\hat{\pi}_j(\mathbf{x}')]=i\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')$. We note that this Hamiltonian expressed classically, with the commutation relations replaced by Poisson bracket relations, gives the correct equations of motion for the vector potential with classical currents. Thus, we have employed the standard procedure to go from a classical problem to a quantum problem. However, this is not the usual method of quantizing an electrodynamic problem and requires further explication. What is unusual is that, since the Coulomb fields of the electron and solenoid pieces are not germane to the AB problem, we just omit them. This gives the treatment of the components of the vector potential as essentially three scalar fields, which simplifies the analysis.

Since the vector potential operator components mutually commute at all locations, we can express the wave function in their eigenbasis. For brevity of notation, we will designate such an eigenvector as $|\mathbf{A}\rangle$, which satisfies $\hat{A}^i(\mathbf{x})|\mathbf{A}\rangle = A^i(\mathbf{x})|\mathbf{A}\rangle$, where the eigenvalues $A^i(\mathbf{x})$ are different functions of $\mathbf{x}$ for each different eigenvector $[-\infty < A^i(\mathbf{x}) < \infty]$. The functional integral will be defined $\int DA \equiv C\prod_{\mathbf{x},i}\int_{-\infty}^{\infty} dA^i(\mathbf{x})$, with C chosen so that $\int DA|\mathbf{A}\rangle\langle\mathbf{A}| = 1$. Of course, this is not the usual quantum electrodynamics: There is no gauge invariance and, in Appendix B where we express the state vector in terms of photons, there are longitudinally polarized photons. In this section we will solve this problem. In Sec. III we will apply this result to an approximate solution of the AB problem.

Wave function

For the classical problem, the Poisson bracket equation of motion for the vector potential that follows from the classical version of the Hamiltonian (3), and its general solution are

$$\frac{\partial^2}{\partial t^2}A^i_{\rm cl}(\mathbf{x},t) = \nabla^2 A^i_{\rm cl}(\mathbf{x},t) + J^i(\mathbf{x},\mathbf{t}), \tag{4a}$$

$$A^i_{\rm cl}(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}[e^{-i\omega t}c^i(\mathbf{k}) + e^{i\omega t}c^{*i}(-\mathbf{k})] + \frac{1}{(2\pi)^3}\int d\mathbf{x}'\int_0^t dt'\, J^i(\mathbf{x}',t')\int d\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\frac{\sin\omega(t-t')}{\omega}, \tag{4b}$$

where $\omega \equiv |\mathbf{k}|$ and the $c^i(\mathbf{k})$ are arbitrary. We are going to solve the quantum problem by assuming a wave function of the form

$$\langle\mathbf{A}|\psi_A,t\rangle = N\exp\left(-\int d\mathbf{x}\, d\mathbf{x}' A^i(\mathbf{x})B(\mathbf{x}-\mathbf{x}',t)A^i(\mathbf{x}') + i\int d\mathbf{x}\, b^i(\mathbf{x},t)A^i(\mathbf{x}) + ic(t)\right). \tag{5}$$

We insert (5) into Schrödinger's equation with the Hamiltonian (3). The algebra is relegated to Appendix A, where it is shown that the solution is

$$\langle\mathbf{A}|\psi_A,t\rangle = N\exp\left(-\int d\mathbf{x}\, d\mathbf{x}'[A^i(\mathbf{x}) - A^i_{\rm cl}(\mathbf{x},t)]\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}[A^i(\mathbf{x}') - A^i_{\rm cl}(\mathbf{x},t)]\right.$$
$$\left. + i\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)[A^i(\mathbf{x}) - A^i_{\rm cl}(\mathbf{x},t)]\right)\exp\left(\frac{i}{2}\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)A^i_{\rm cl}(\mathbf{x},t) + \frac{i}{2}\int_0^t dt'\int d\mathbf{x}\, A^i_{\rm cl}(\mathbf{x},t')J^i(\mathbf{x},t')\right). \tag{6}$$

It is worth remarking that there is no free parameter allowing adjustment of the width of $|\langle\mathbf{A}|\psi_A,t\rangle|$, unlike, e.g., the case for the initial state of a particle, $\langle x|\psi\rangle \sim e^{-[(x-x_0)^2/\sigma^2]}$. However, an adjustable width σ appears if instead of a point electron current source for the vector potential, the electron charge density is smeared over a distance σ. In the next section, such a smearing is shown to be necessary.

In Appendix B it is shown that this state vector can be expressed in terms of coherent states of photons

$$|\psi_A,t\rangle = \exp\left(\int d\mathbf{k}\,\alpha^i(\mathbf{k},t)a^{i\dagger}(\mathbf{k},t)\right)|0\rangle\exp\left(-\frac{1}{2}\int d\mathbf{k}|\alpha^i(\mathbf{k},t)|^2\right)\exp\left(\frac{i}{2}\int_0^T dt\int d\mathbf{x}\, A^i_{\rm cl}(\mathbf{x},t)J^i(\mathbf{x},t)\right), \tag{7}$$

where

$$\alpha^i(\mathbf{k},T) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\sqrt{\frac{\omega}{2}}A^i_{\rm cl}(\mathbf{x},t) + i\frac{1}{\sqrt{2\omega}}\dot{A}^i_{\rm cl}(\mathbf{x},t)\right] \tag{8}$$

and, as usual, $\hat{A}^i(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{k}}{\sqrt{2\omega}}[e^{i\mathbf{k}\cdot\mathbf{x}}a^i(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x}}a^{i\dagger}(\mathbf{k})]$. In what follows we will only utilize Eq. (6).

III. THE AB PHASE OF THE VECTOR POTENTIAL WAVE FUNCTION

We now turn to apply Eq. (6) to the special case of the AB situation. The wave function for the AB effect is the sum of two wave functions, one for the right traverse of the electron and one for the left traverse of the electron. In the approximation considered here, the only interaction is that of the quantized vector potential interacting with the classical electron and solenoid currents. These classical currents are $\mathbf{J}_{\rm sol}(\mathbf{x})$, the constant current for the solenoid, and $\mathbf{J}_{\rm el}(\mathbf{x},t)$, the current of a classical electron orbiting the solenoid with speed u at radius R in a half circle (counterclockwise for the right traverse and clockwise for the left traverse). Then the wave function for each traverse is the product of three independently evolving wave functions: those of the vector potential, electron, and solenoid. The solenoid wave function is the same for either traverse. The electron wave packets produces no phase of their own that differs for the two traverses. Thus, we are considering a wave function of the form

$$|\psi,t\rangle = \frac{1}{\sqrt{2}}[|\psi_R,t\rangle + |\psi_L,t\rangle]$$
$$= \frac{1}{\sqrt{2}}|\psi_{\rm sol},t\rangle[|\psi_{{\rm el},R},t\rangle|\psi_{A,R},t\rangle + |\psi_{{\rm el},L},t\rangle|\psi_{A,L},t\rangle],$$

where the Schrödinger equation for the right traverse satisfies this approximation to Eq. (1):

$$i\frac{d}{dt}|\psi_R,t\rangle = \left[\hat{H}_{\rm el} + \hat{H}_{\rm sol} + \hat{H}_A - \int d\mathbf{x}[\mathbf{J}_{el,R}(\mathbf{x}) + \mathbf{J}_{\rm sol}(\mathbf{x})]\cdot\hat{\mathbf{A}}(\mathbf{x})\right]|\psi_R,t\rangle$$

(and similarly for the left traverse).

After completion of the traverses of the packets at time $T = \pi R/u$, the two electron packets are presumed to meet the electron equivalent of a half-silvered mirror, resulting in $\frac{1}{\sqrt{2}}$ times the sum of packets emerging from one side and $\frac{1}{\sqrt{2}}$ times the difference emerging from the other. Since neither the electron nor the solenoid contributes a phase shift, the probabilities of detecting the electron on one side or the other are $\frac{1}{2}[1 \pm e^{-a(T)} \cos \Phi(T)]$, where $e^{-a(t)} e^{i\Phi(t)} = \langle \psi_{A,L},t|\psi_{A,R},t\rangle$.

It is shown in Appendix C that the first exponential in Eq. (6) produces no phase shift. Thus, the second exponent is responsible for the phase (the difference of which for the two trajectories gives the phase shift).

We will look at that phase in a moment but first it is also shown in Appendix C that with the classical electron current that of a point particle, there is an ultraviolet (short distance) divergence causing $a(T) = \infty$. However, it is also shown, with the electron's charge density smeared over a length the size σ of an electron wave packet, that the integral is finite, with $a(T) \sim \frac{e^2}{\hbar c}\frac{u}{c}\frac{R}{\sigma} \ll 1$; note the telltale σ^{-1} dependence. (We may consider $\frac{R}{\sigma} \approx 1$, which was the case in the experiment [1] where the packets traveling through and outside the magnetized torus were the order of the torus size.) With $e^{-a(T)} \approx 1$, there is maximum interference.

Return now to the phase in the second line of (6) and, to be concrete, suppose we evaluate it at time T at the end of the electron's right traverse:

$$\begin{aligned}\Phi(T) &\equiv \frac{1}{2}\int d\mathbf{x}\, \dot{A}^i_{\rm cl}(\mathbf{x},T)A^i_{\rm cl}(\mathbf{x},T)\\ &\quad+\frac{1}{2}\int_0^T dt'\int d\mathbf{x}\, A^i_{\rm cl}(\mathbf{x},t')J^i(\mathbf{x},t')\\ &\equiv \Phi_1(T)+\Phi_2(T). \end{aligned} \tag{9}$$

Consider $\Phi_2(T)$ first. We substitute $A^i_{\rm cl}(\mathbf{x},t') = A^i_{\rm el}(\mathbf{x},t') + A^i_{\rm sol}(\mathbf{x})$ and $J^i(\mathbf{x},t') = J^i_{\rm el}(\mathbf{x},t') + J^i_{\rm sol}(\mathbf{x})$ and drop terms $A^i_{\rm el}(\mathbf{x},t')J^i_{\rm el}(\mathbf{x},t')$ and $A^i_{\rm sol}(\mathbf{x},t')J^i_{\rm sol}(\mathbf{x})$, which are the scalar product of two terms with the same subscript, since they are the same for both traverses of the electron:[1]

$$\begin{aligned}\Phi_2(T) &\equiv \Phi_{21}(T)+\Phi_{22}(T)\\ &\equiv \frac{1}{2}\int d\mathbf{x}\int_0^T dt'\big[J^i_{\rm el}(\mathbf{x},t')A^i_{\rm sol}(\mathbf{x}) + A^i_{\rm el}(\mathbf{x},t')J^i_{\rm sol}(\mathbf{x})\big]. \end{aligned} \tag{10}$$

Thus $\Phi_2(T)$ is 1/2 the sum of a phase due to the electron moving in the vector potential of the solenoid and a phase due to the solenoid moving in the vector potential of the electron.

The integral in the first term is the well-known contribution, to the usual AB phase shift Φ_{AB}, of the right traverse of the electron:

$$\begin{aligned}2\Phi_{21}(T) &= \int d\mathbf{x}\int_0^T dt'\, J^i_{\rm el}(\mathbf{x},t')A^i_{\rm sol}(\mathbf{x})\\ &= \int_0^T dt' e\mathbf{u}_{\rm el}(t')\cdot A^i_{\rm sol}[\mathbf{x}_{\rm el}(t')]\\ &= \int_0^{\mathbf{x}_{\rm el}(T)} e\, d\mathbf{x}_{\rm el}\cdot A^i_{\rm sol}(\mathbf{x}_{\rm el}) = \frac{1}{2}\Phi_{AB}, \end{aligned} \tag{11}$$

where we have used $J^i_{\rm el}(\mathbf{x},t') = e\mathbf{u}_{\rm el}(t')\delta(\mathbf{x}-\mathbf{x}_{\rm el}(t'))$ and $\mathbf{u}_{\rm el}(t')dt' = d\mathbf{x}_{\rm el}$. The integral in the second term $2\Phi_{22}(T)$ was shown in [6] to be the phase contributed by the solenoid charges moving under the vector potential of the right traverse of the electron. In the nonrelativistic approximation, where $\nabla^2 A^i_{\rm el}(\mathbf{x},t) \approx -J^i_{\rm el}(\mathbf{x},t)$, this is equal to the right traverse's contribution to the AB phase shift:

$$\begin{aligned}2\Phi_{22}(T) &= \int d\mathbf{x}\int_0^T dt'\, A^i_{\rm el}(\mathbf{x},t')J^i_{\rm sol}(\mathbf{x})\\ &= \int d\mathbf{x}\int_0^T dt'\, A^i_{\rm el}(\mathbf{x},t')(-)\nabla^2 A^i_{\rm sol}(\mathbf{x})\\ &= \int d\mathbf{x}\int_0^T dt'(-)\nabla^2 A^i_{\rm el}(\mathbf{x},t')A^i_{\rm sol}(\mathbf{x})\\ &= \int d\mathbf{x}\int_0^T dt'\, J^i_{\rm el}(\mathbf{x},t')A^i_{\rm sol}(\mathbf{x})\\ &= 2\Phi_{21}(T) = \frac{1}{2}\Phi_{AB}. \end{aligned} \tag{12}$$

However, here, a relativistic calculation is being done. Therefore, we must use $\nabla^2 A^i_{\rm el}(\mathbf{x},t) - \ddot{A}^i_{\rm el}(\mathbf{x},t) = -J^i_{\rm el}(\mathbf{x},t)$. To get the same result as (12) requires the inclusion of $\Phi_1(T)$. We now show that $2\Phi_{22}(T)$ and $2\Phi_1(T)$ combine to give $\frac{1}{2}\Phi_{AB}$.

The first three steps of (12) are valid:

$$\begin{aligned}2\Phi_{22}(T) &\equiv \int d\mathbf{x}\int_0^T dt'\, A^i_{\rm el}(\mathbf{x},t')J^i_{\rm sol}(\mathbf{x})\\ &= -\int d\mathbf{x}\int_0^T dt'\, A^i_{\rm el}(\mathbf{x},t')\nabla^2 A^i_{\rm sol}(\mathbf{x})\\ &= -\int d\mathbf{x}\int_0^T dt'\,\nabla^2 A^i_{\rm el}(\mathbf{x},t')A^i_{\rm sol}(\mathbf{x}). \end{aligned} \tag{13}$$

Now $\Phi_1(T)$, although evaluated at time T, actually depends upon the time interval $(0,T)$. We note that $\dot{A}^i_{\rm cl}(\mathbf{x},t) = d[A^i_{\rm sol}(\mathbf{x}) + A^i_{\rm el}(\mathbf{x},t)] = \dot{A}^i_{\rm el}(\mathbf{x},t)$. When the electron accelerates, it begins to radiate and spheres of vector potential move out with the speed of light from each point on its trajectory, so $\dot{A}^i_{\rm el}(\mathbf{x},t)$ is nonzero throughout the radiated region.

Disregarding $\int d\mathbf{x}\, \dot{A}^i_{\rm el}(\mathbf{x},T)A^i_{\rm el}(\mathbf{x})$ as it is the same for both traverses of the electron, we have

$$\begin{aligned}2\Phi_1(T) &= \int d\mathbf{x}\, \dot{A}^i_{\rm el}(\mathbf{x},T)A^i_{\rm sol}(\mathbf{x})\\ &= \int d\mathbf{x}\int_0^T dt'\frac{\partial}{\partial t'}\big[\dot{A}^i_{\rm el}(\mathbf{x},t')A^i_{\rm sol}(\mathbf{x})\big]\\ &= \int d\mathbf{x}\int_0^T dt'\left[\frac{\partial^2}{\partial t'^2}A^i_{\rm el}(\mathbf{x},t')\right]A^i_{\rm sol}(\mathbf{x}) \end{aligned} \tag{14}$$

[we have assumed that the electron is at rest at time 0+ and accelerates immediately thereafter, so the boundary value

[1]This is obvious for $\mathbf{A}_{\rm sol}\cdot\mathbf{J}_{\rm sol}$. For the electron, the two trajectories are exchanged under a rotation of 180° about the y axis, so $\int d\mathbf{x}\, A^i_{\rm el}(\mathbf{x},t')J^i_{\rm el}(\mathbf{x},t')$ is unchanged.

$\dot{A}^i_{\rm el}(\mathbf{x},0) = 0$]. Therefore,

$$\begin{aligned} &2\Phi_1(T) + 2\Phi_{22}(T) \\ &= \int d\mathbf{x} \int_0^T dt' \left[\frac{\partial^2}{\partial t'^2} A^i_{\rm el}(\mathbf{x},t') - \nabla^2 A^i_{\rm el}(\mathbf{x},t') \right] A^i_{\rm sol}(\mathbf{x}) \\ &= \int d\mathbf{x} \int_0^T dt' J^i_{\rm el}(\mathbf{x},t') A^i_{\rm sol}(\mathbf{x}) = \frac{1}{2}\Phi_{AB}. \end{aligned} \quad (15)$$

Thus, the electron's right traverse produces the phase $\Phi(T) = \Phi_{21}(T) + [\Phi_1(T) + \Phi_{22}(T)] = \frac{1}{4}\Phi_{AB} + \frac{1}{4}\Phi_{AB} = \frac{1}{2}\Phi_{AB}$ and the left traverse produces the phase $-\frac{1}{2}\Phi_{AB}$, which is to be subtracted, giving the total Φ_{AB}. This concludes our demonstration that the AB phase shift is obtained from the phase of the wave function describing the quantized vector potential with classical electron and solenoid current sources.

IV. APPROXIMATE SOLUTION FOR THE EXACT AB PROBLEM

For our next enterprise, we turn to approximating the solution of the exact problem, Schrödinger's equation (1). Consider the variational problem $\delta I = 0$, where

$$I \equiv \int_0^T dt \left[\langle\psi,t| i\frac{d}{dt}|\psi,t\rangle - \langle\psi,t|\hat{H}|\psi,t\rangle \right]. \quad (16)$$

The variation with respect to $\langle\psi,t|$ gives the Schrödinger equation and the independent variation of $|\psi,t\rangle$ gives its Hermitian conjugate.

We consider a situation where the electron and solenoid particles are well localized. In that circumstance, for either the left or right traverse of the electron around the solenoid, we approximate the actual state vector by a direct product of all three involved entities: $|\Psi,t\rangle \approx |\psi_A,t\rangle|\psi_{\rm el},t\rangle|\psi_{\rm sol},t\rangle$. Also, we consider that well localized means that the expectation value of the current operators are well approximated by the corresponding classical currents: $\langle\psi_{\rm el},t|\hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi_{\rm el},t\rangle \approx \mathbf{J}_{\rm el}(\mathbf{x},t)$, $\langle\psi_{\rm sol},t|\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi_{\rm sol},t\rangle \approx \mathbf{J}_{\rm sol}(\mathbf{x},t)$, and $\mathbf{J}(\mathbf{x},t) \equiv \mathbf{J}_{\rm el}(\mathbf{x},t) + \mathbf{J}_{\rm sol}(\mathbf{x},t)$.

A. Schrödinger equations

Upon substituting $|\Psi,t\rangle \approx |\psi_A,t\rangle|\psi_{\rm el},t\rangle|\psi_{\rm sol},t\rangle$ into (16) with $\hat{H}$ given by (1) and varying $\langle\psi_A,t|$, $\langle\psi_{\rm el},t|$, and $\langle\psi_{\rm sol},t|$ independently, we obtain three coupled equations

$$\begin{aligned} &i\frac{d}{dt}|\psi_A,t\rangle + \left\{ \langle\psi_{\rm el},t| \left[i\frac{d}{dt} - \hat{H}_{\rm el} \right] |\psi_{\rm el},t\rangle + \langle\psi_{\rm sol},t| \left[i\frac{d}{dt} - \hat{H}_{\rm sol} \right] |\psi_{\rm sol},t\rangle \right\} |\psi_A,t\rangle \\ &= \hat{H}_A|\psi_A,t\rangle - \int d\mathbf{x} [\langle\psi_{\rm el},t|\hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi_{\rm el},t\rangle + \langle\psi_{\rm sol},t|\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi_{\rm sol},t\rangle] \cdot \hat{\mathbf{A}}(\mathbf{x},t)|\psi_A,t\rangle \approx [\hat{H}_A - \mathbf{J}(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)]|\psi_A,t\rangle, \end{aligned} \quad (17a)$$

$$\begin{aligned} &i\frac{d}{dt}|\psi_{\rm el},t\rangle + \left\{ \langle\psi_A,t| \left[i\frac{d}{dt} - \hat{H}_A \right] |\psi_A,t\rangle + \langle\psi_{\rm sol},t| \left[i\frac{d}{dt} - \hat{H}_{\rm sol} \right] |\psi_{\rm sol},t\rangle \right. \\ &\left. + \int d\mathbf{x} \langle\psi_A,t|\hat{\mathbf{A}}(\mathbf{x},t)|\psi_A,t\rangle \cdot \langle\psi_{\rm sol},t|\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi_{\rm sol},t\rangle \right\} |\psi_{\rm el},t\rangle \\ &= \hat{H}_{\rm el}|\psi_{\rm el},t\rangle - \int d\mathbf{x} \langle\psi_A,t|\hat{\mathbf{A}}(\mathbf{x},t)|\psi_A,t\rangle \cdot \hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi_{\rm el},t\rangle = \hat{H}_{\rm el}|\psi_{\rm el},t\rangle - \int d\mathbf{x}\, \mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi_{\rm el},t\rangle, \end{aligned} \quad (17b)$$

$$\begin{aligned} &i\frac{d}{dt}|\psi_{\rm sol},t\rangle + \left\{ \langle\psi_A,t| \left[i\frac{d}{dt} - \hat{H}_A \right] |\psi_A,t\rangle + \langle\psi_{\rm el},t| \left[i\frac{d}{dt} - \hat{H}_{\rm el} \right] |\psi_{\rm el},t\rangle \right. \\ &\left. + \int d\mathbf{x} \langle\psi_A,t|\hat{\mathbf{A}}(\mathbf{x},t)|\psi_A,t\rangle \cdot \langle\psi_{\rm el},t|\hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi_{\rm el},t\rangle \right\} |\psi_{\rm sol},t\rangle \\ &= \hat{H}_{\rm sol}|\psi_{\rm sol},t\rangle - \int d\mathbf{x} \langle\psi_A,t|\hat{\mathbf{A}}(\mathbf{x},t)|\psi_A,t\rangle \cdot \hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi_{\rm sol},t\rangle = \hat{H}_{\rm sol}|\psi_{\rm sol},t\rangle - \int d\mathbf{x}\, \mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi_{\rm sol},t\rangle. \end{aligned} \quad (17c)$$

We have anticipated that the equations are of Hamiltonian form, so the time evolution is unitary and we may take $\langle\psi_A,t|\psi_A,t\rangle = \langle\psi_{\rm el,sol},t|\psi_{\rm el,sol},t\rangle = 1$. We have used the fact (from Eq. (6)) that $\langle\psi_A,t|\hat{\mathbf{A}}(\mathbf{x},t)|\psi_A,t\rangle = \mathbf{A}_{\rm cl}(\mathbf{x},t)$ for this state vector, in the last line of Eqs. (17b) and (17c).

Equation (17a) is the equation we have considered, in Secs. II and III, for the quantized vector potential with classical electron and solenoid sources *except* for the additional c-number term in the curly brackets. That term is real (take the complex conjugate and use $d\langle\psi_{\rm el},t|\psi_{\rm el},t\rangle/dt = d\langle\psi_{\rm sol},t|\psi_{\rm sol},t\rangle/dt = 0$), so it is the time derivative of a phase. Without that term, we have found the phase of the vector potential wave function. We have seen that, in the magnetic AB situation, for, e.g., the right traverse of the electron, it gives 1/2 the AB phase shift.

Equation (17b) is the equation for the quantized electron with a classical vector potential due to the solenoid (the classical vector potential part due to the electron itself may be ignored, as this self-interaction phase contribution is the same for left and right traverses of the electron) *except* for the additional phase term in the curly brackets. Without that term, the phase of the electron wave function, for the right traverse of the electron, gives of course 1/2 the AB phase shift.

Equation (17c) is the equation for the quantized solenoid with a classical vector potential due to the electron (the classical vector potential part due to the solenoid itself may be ignored, as this self-interaction phase contribution is the same for left and right traverses of the electron) except for the additional phase term in the curly brackets. Without that term, the phase of the solenoid wave function, for the right traverse of the electron, was shown in [6] and in the preceding section that it gives 1/2 the AB phase shift.

B. Extra phase

Now we note that adding a phase factor to each state vector does not change the phase of the overall state vector provided the phases sum to zero. We will use this to eliminate the extra phase term from the electron and solenoid Schrödinger equations (although we could have made a different choice).

Given any solution for the three state vectors, write $|\psi_{\rm el},t\rangle=|\psi'_{\rm el},t\rangle e^{i\phi_{\rm el}(t)}$, $|\psi_{\rm sol},t\rangle=|\psi'_{\rm sol},t\rangle e^{i\phi_{\rm sol}(t)}$, and $|\psi_A,t\rangle=|\psi'_A,t\rangle e^{-i[\phi_{\rm el}(t)+\phi_{\rm sol}(t)]}$, where $\dot\phi_{\rm el}(t)$ is the curly bracket term in the electron's Schrödinger equation and $\dot\phi_{\rm sol}(t)$ is the curly bracket term in the solenoid's Schrödinger equation. Then $|\Psi,t\rangle=|\psi_A,t\rangle|\psi_{\rm el},t\rangle|\psi_{\rm sol},t\rangle=|\psi'_A,t\rangle|\psi'_{\rm el},t\rangle|\psi'_{\rm sol},t\rangle$ and we get [note that in (17a), when the substitution of unprimed for primed state vectors is made and the time derivatives of the phase factors are taken, $\dot\phi_{\rm el}(t)$ and $\dot\phi_{\rm sol}(t)$ produce no net contribution]

$$i\frac{d}{dt}|\psi'_A,t\rangle+\left\{\langle\psi'_{\rm el},t|\left[i\frac{d}{dt}-\hat H_{\rm el}\right]|\psi'_{\rm el},t\rangle+\langle\psi'_{\rm sol},t|\left[i\frac{d}{dt}-\hat H_{\rm sol}\right]|\psi'_{\rm sol},t\rangle\right\}\psi'_A,t\rangle=[\hat H_A-\mathbf{J}(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)]|\psi'_A,t\rangle, \tag{18a}$$

$$i\frac{d}{dt}|\psi'_{\rm el},t\rangle=\hat H_{\rm el}|\psi'_{\rm el},t\rangle-\int d\mathbf{x}\,\mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi'_{\rm el},t\rangle, \tag{18b}$$

$$i\frac{d}{dt}|\psi'_{\rm sol},t\rangle=\hat H_{\rm sol}|\psi'_{\rm sol},t\rangle-\int d\mathbf{x}\,\mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi'_{\rm sol},t\rangle. \tag{18c}$$

The extra phase terms in (18a) are obtained from (18b) and (18c) by taking the expectation values

$$\langle\psi'_{\rm el},t|\left[i\frac{d}{dt}-\hat H_{\rm el}\right]|\psi'_{\rm el},t\rangle=-\langle\psi'_{\rm el},t|\int d\mathbf{x}\,\mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\hat{\mathbf{J}}_{\rm el}(\mathbf{x})|\psi'_{\rm el},t\rangle=-\int d\mathbf{x}\,\mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\mathbf{J}_{\rm el}(\mathbf{x},t), \tag{19a}$$

$$\langle\psi'_{\rm sol},t|\left[i\frac{d}{dt}-\hat H_{\rm sol}\right]|\psi'_{\rm sol},t\rangle=-\langle\psi'_{\rm sol},t|\int d\mathbf{x}\,\mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\hat{\mathbf{J}}_{\rm sol}(\mathbf{x})|\psi'_{\rm sol},t\rangle=-\int d\mathbf{x}\,\mathbf{A}_{\rm cl}(\mathbf{x},t)\cdot\mathbf{J}_{\rm sol}(\mathbf{x},t). \tag{19b}$$

The right-hand sides of (19a) and (19b) are the same, e.g., for the electron's right traverse, both equal to $-\dot\Phi(t)$, where $\Phi(T)=\frac{1}{2}\Phi_{AB}$. Thus, (18a) becomes

$$i\frac{d}{dt}|\psi'_A,t\rangle-2\dot\Phi(t)=[\hat H_A-\mathbf{J}(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)]|\psi'_A,t\rangle. \tag{20}$$

For its right traverse (a similar result holds for the left traverse), the phase contribution of the electron wave function satisfying (18b) is $e^{i\Phi(t)}$, as is the phase contribution of the solenoid wave function satisfying (18c). The phase contribution of the vector potential wave function satisfying (18a) *without* the extra phase term is also $e^{i\Phi(t)}$, but with the extra phase terms it is $e^{-i\Phi(t)}$, Therefore, the combined phase is $e^{i\Phi(t)}$ and at time T this is the correct phase contribution to the AB phase shift, $e^{i(1/2)\Phi_{AB}}$.

V. CONCLUDING REMARKS

A well-trained physicist is supposed to know intuitively how to split the world into classical and quantum, in order to do theoretical analysis, of the truly quantum acting in a classical background. The magnetic AB effect appears to provide an example where one does not need to be well trained. For the interaction, the split of electron, solenoid, and vector potential contributions into two classical and one quantum may be made any way and one gets the right answer. A contribution of this paper has been to show that the quantized vector potential is an equal partner in this troika.

Why this works is not intuitively clear. Reading from left to right, if asked to put in order what is most quantum (least classical) to most classical (least quantum), most people would make the following list: electron, vector potential, solenoid. Then, why should it be that the current of the most quantum thing, the electron, can blithely be made classical in the interaction in two of the three calculations?

This paper provides a cautionary tale. Since the whole world is quantum, the only sure thing is to make all three entities quantum. Then one can try to proceed from there to make justifiable approximations.

Surely, one would think, if it apparently is a good approximation (since one gets the right answer) to make any two things in the interaction classical, it surely should work just fine, even be a better approximation, to make just one of the two things classical. However, we have seen that is not the case. Just replacing the vector potential operator by its classical counterpart in the interaction gives the wrong answer.

What goes wrong with this naive approach is that it amounts to simply adding the phase shifts. This arises because the Hamiltonian, as shown in (2), is then separable and thus the wave function is simply the product of the two wave functions for the separate parts of the compound system.

In the preceding paper [6] we show that a more careful approximation, while not employing the quantized vector potential field, is to characterize the interaction using the vector potential expressed as a function of the position and momentum operators associated with the electron and solenoid particles. Then, while the wave function is again approximated as the product of the electron and solenoid wave functions, a variational approximation just like the one used in this paper shows that an additional phase shift must be included for the system as a whole, and this is essential to reconciling the results.

A second contribution of this paper has been to show, similarly, how the magnetic AB phase shift (highly similar considerations occur for the electric AB effect [6]) arises when all three quantizable entities are treated on an equal basis. It is clear, within the framework of our discussion, that there is no way to prefer one split of the world over another, to prefer the notion that the phase shift is due to the electron's motion in the solenoid's vector potential, over that it is due to the solenoid particles' motion in the electron's vector potential, over that it is due to the vector potential evolving governed by the sources of electron and solenoid currents.

Presumably the exact solution where all three entities are quantized would give the AB shift. One surmises it would not be possible to attribute it to any one entity, just as is the case here, with the approximation to the exact solution. This example may be considered to illustrate the holism of quantum theory's description of nature.

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APPENDIX A: CALCULATION OF THE WAVE FUNCTION FOR THE VECTOR POTENTIAL WITH A CLASSICAL CURRENT SOURCEl

We start with the wave function (5),

$$\langle\mathbf{A}|\psi_A,t\rangle = N\exp\left(-\int d\mathbf{x}\,d\mathbf{x}'A^i(\mathbf{x})B(\mathbf{x}-\mathbf{x}',t)A^i(\mathbf{x}')+i\int d\mathbf{x}\,b^i(\mathbf{x},t)A^i(\mathbf{x})+ic(t)\right), \tag{A1}$$

where B is symmetrical in its argument, $B(\mathbf{x}-\mathbf{x}',t)=B(\mathbf{x}'-\mathbf{x},t)$. We insert it into the Schrödinger equation with the Hamiltonian (3):

$$i\frac{d}{dt}\langle\mathbf{A}|\psi_A,t\rangle = \int d\mathbf{x}\left[\frac{1}{2}\left(-\frac{\delta^2}{\delta A^{i2}(\mathbf{x})}+\nabla A^i(\mathbf{x})\cdot\nabla A^i(\mathbf{x})\right)-J^i(\mathbf{x},t)A^i(\mathbf{x})\right]\langle\mathbf{A}|\psi_A,t\rangle. \tag{A2}$$

Evaluating both sides of (A2) and dividing by $\langle\mathbf{A}|\psi,t\rangle$, we have

$$\begin{aligned}
&-\int d\mathbf{x}\,d\mathbf{x}'A^i(\mathbf{x})i\frac{\partial}{\partial t}B(\mathbf{x}-\mathbf{x}',t)A^i(\mathbf{x}')-\int d\mathbf{x}\frac{\partial}{\partial t}b^i(\mathbf{x},t)A^i(\mathbf{x})-\frac{\partial}{\partial t}c(t)\\
&=\int d\mathbf{x}\left\{\frac{1}{2}\left[-\left(-2\int d\mathbf{x}'B(\mathbf{x}-\mathbf{x}',t)A^i(\mathbf{x}')+ib^i(\mathbf{x},t)\right)^2+2B(0,t)\right]+\frac{1}{2}\nabla A^i(\mathbf{x})\cdot\nabla A^i(\mathbf{x})-J^i(\mathbf{x},t)A^i(\mathbf{x})\right\}\\
&=-2\int d\mathbf{x}_1\int d\mathbf{x}\int d\mathbf{x}'B(\mathbf{x}_1-\mathbf{x},t)B(\mathbf{x}_1-\mathbf{x}',t)A^i(\mathbf{x})A^i(\mathbf{x}')+\frac{1}{2}\int d\mathbf{x}\int d\mathbf{x}'\delta(\mathbf{x}-\mathbf{x}')\nabla A^i(\mathbf{x})\cdot\nabla A^i(\mathbf{x}')\\
&\quad+2i\int d\mathbf{x}\int d\mathbf{x}'B(\mathbf{x}-\mathbf{x}',t)A^i(\mathbf{x}')b^i(\mathbf{x},t)-\int d\mathbf{x}\,J^i(\mathbf{x},t)A^i(\mathbf{x})+\frac{1}{2}\int d\mathbf{x}\,b^{i2}(\mathbf{x},t).
\end{aligned} \tag{A3}$$

In the last step, we have dropped the phase $\int d\mathbf{x}\,B(0,t)$.

Vanishing of the coefficients of $A^i(\mathbf{x})A^i(\mathbf{x}')$, $A^i(\mathbf{x})$, and 1 implies the three conditions

$$-i\dot{B}(\mathbf{x}-\mathbf{x}',t) = -2\int d\mathbf{x}_1 B(\mathbf{x}-\mathbf{x}_1,t)B(\mathbf{x}'-\mathbf{x}_1,t)+\frac{1}{2}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{x}'}\delta(\mathbf{x}-\mathbf{x}'), \tag{A4a}$$

$$-\dot{b}^i(\mathbf{x},t) = 2i\int d\mathbf{x}'B(\mathbf{x}-\mathbf{x}',t)b^i(\mathbf{x}',t)-J^i(\mathbf{x},t), \tag{A4b}$$

$$-\dot{c} = \frac{1}{2}\int d\mathbf{x}'b^{i2}(\mathbf{x}',t). \tag{A4c}$$

Equation (A4a) is a generalized form of the Riccati equation and the following general solution may readily be verified:

$$\begin{aligned}
B(\mathbf{x}-\mathbf{x}',t) &= \frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\frac{-f(\mathbf{k})e^{-i\omega t}+e^{i\omega t}}{f(\mathbf{k})e^{-i\omega t}+e^{i\omega t}}\\
&= \frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\frac{1-|f(\mathbf{k})|^2-f(\mathbf{k})e^{-i2\omega t}+f^*(\mathbf{k})e^{2i\omega t}}{|f(\mathbf{k})e^{-i\omega t}+e^{i\omega t}|^2},
\end{aligned} \tag{A5}$$

where $f(\mathbf{k})$ is a symmetric $[f(\mathbf{k})=f(-\mathbf{k})]$ function [since $B(\mathbf{x}-\mathbf{x}',t)$ is symmetric], but otherwise arbitrary. In order that $|\langle\mathbf{A}|\psi,t\rangle|^2$ be integrable, it is necessary that the quadratic form in its exponent [found from (A1) and (A5) and noting that

$-f(\mathbf{k})e^{-i2\omega t}+f^*(\mathbf{k})e^{2i\omega t}$ is imaginary],

$$\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega\left|\int d\mathbf{x}\,A^i(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2\frac{1-|f(\mathbf{k})|^2}{|f(\mathbf{k})e^{-i\omega t}+e^{i\omega t}|^2},$$

be positive definite, so we must have $|f(\mathbf{k})|<1$. In order to have a time-translationally invariant solution, we must make the coherent state choice $f(\mathbf{k})=0$, so we arrive at

$$B(\mathbf{x}-\mathbf{x}')=\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{A6}$$

Equation (A4b) may now be solved, with the result

$$b^i(\mathbf{x},t)=\frac{1}{(2\pi)^3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega t}g^i(\mathbf{k})+\frac{1}{(2\pi)^3}\int d\mathbf{x}'\int_0^t dt'\,J^i(\mathbf{x}',t')\int d\mathbf{k}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')}, \tag{A7}$$

where $g^i(\mathbf{k})$ is an arbitrary function. Further, $ib^i(\mathbf{x},t)$, separated into real and imaginary parts, becomes

$$\begin{aligned}ib^i(\mathbf{x},t)=&\frac{i}{2(2\pi)^3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{x}}[e^{-i\omega t}g^i(\mathbf{k})-e^{i\omega t}g^{i*}(-\mathbf{k})]\\&+\frac{1}{(2\pi)^3}\int d\mathbf{x}'\int_0^t dt'\,J^i(\mathbf{x}',t')\int d\mathbf{k}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\sin\omega(t-t')\\&+\frac{i}{2(2\pi)^3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{x}}[e^{-i\omega t}g^i(\mathbf{k})+e^{i\omega t}g^{i*}(-\mathbf{k})]\\&+\frac{i}{(2\pi)^3}\int d\mathbf{x}'\int_0^t dt'\,J^i(\mathbf{x}',t')\int d\mathbf{k}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\cos\omega(t-t').\end{aligned} \tag{A8}$$

This can be expressed in terms of the notation in the classical solution (4b) by writing $c^i(\mathbf{k})=\frac{i}{2\omega}g^i(\mathbf{k})$, obtaining

$$ib^i(\mathbf{x},t)=2\int d\mathbf{x}'B(\mathbf{x}-\mathbf{x}')A^i_{\rm cl}(\mathbf{x}',t)+i\dot{A}^i_{\rm cl}(\mathbf{x},t). \tag{A9}$$

We will shortly see the exponent in the solution is then a quadratic form in $A^i(\mathbf{x})-A^i_{\rm cl}(\mathbf{x},t)$, i.e., the mean value of the vector potential is the classical value.

Equation (A4c) may now be written as

$$\begin{aligned}ic(t)&=-\frac{i}{2}\int_0^t dt'\int d\mathbf{x}\,b^{i2}(\mathbf{x},t')\\&=-\frac{i}{2}\int_0^t dt'\int d\mathbf{x}\Big[\dot{A}^{i2}_{\rm cl}(\mathbf{x},t')-4\Big(\int d\mathbf{x}'B(\mathbf{x}-\mathbf{x}')A^i_{\rm cl}(\mathbf{x}',t')\Big)^2\Big]\\&\quad-2\int_0^t dt'\int d\mathbf{x}\,d\mathbf{x}'\dot{A}^i_{\rm cl}(\mathbf{x},t')B(\mathbf{x}-\mathbf{x}')A^i_{\rm cl}(\mathbf{x}',t').\end{aligned} \tag{A10}$$

The last (real) term of (A10) may be written as

$$-\int_0^t dt'\frac{d}{dt'}\int d\mathbf{x}\,d\mathbf{x}'\,A^i_{\rm cl}(\mathbf{x},t')B(\mathbf{x}-\mathbf{x}')A^i_{\rm cl}(\mathbf{x}',t')=-\int d\mathbf{x}\,d\mathbf{x}'A^i_{\rm cl}(\mathbf{x},t)B(\mathbf{x}-\mathbf{x}')A^i_{\rm cl}(\mathbf{x}',t)+C. \tag{A11}$$

The constant term e^C may be absorbed into the normalization constant N and so removed from consideration.

In the second (imaginary) term of Eq. (A10), we note, using (A6) [or (A4a) since $\dot{B}=0$], that

$$\int d\mathbf{x}_1\,B(\mathbf{x}_1-\mathbf{x})B(\mathbf{x}_1-\mathbf{x}')=\frac{1}{4(2\pi)^3}\int d\mathbf{k}\,\omega^2e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}=\frac{1}{4}\nabla_x\cdot\nabla_{x'}\delta(\mathbf{x}-\mathbf{x}'). \tag{A12}$$

Using (A6) and (A9)–(A12) and adding and subtracting $i\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)A^i_{\rm cl}(\mathbf{x},t)$, we find that the wave function (A1) satisfying Schrödinger's equation is

$$\begin{aligned}\langle\mathbf{A}|\psi_A,t\rangle=&N\exp\Big(-\int d\mathbf{x}\,d\mathbf{x}'\big[A^i(\mathbf{x})-A^i_{\rm cl}(\mathbf{x},t)\big]\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\big[A^i(\mathbf{x}')-A^i_{\rm cl}(\mathbf{x},t)\big]\\&+i\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)\big[A^i(\mathbf{x})-A^i_{\rm cl}(\mathbf{x},t)\big]\Big)\exp\Big(i\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)A^i_{\rm cl}(\mathbf{x},t)\\&-\frac{i}{2}\int_0^t dt'\int d\mathbf{x}\big[\dot{A}^{i2}_{\rm cl}(\mathbf{x},t')-\nabla A^i_{\rm cl}(\mathbf{x},t')\cdot\nabla A^i_{\rm cl}(\mathbf{x},t')\big]\Big).\end{aligned} \tag{A13}$$

Write the two terms in the phase in (A13) as $\Phi(t) = \Phi_1(t) + \Phi_2(t)$, where $\Phi_2(t)$ may be further simplified. Integrate both terms in the square bracket by parts, the first with respect to time and the second with respect to space:

$$\begin{aligned}\Phi_2(t) &\equiv -\frac{1}{2}\int_0^t dt' \int d\mathbf{x}\big[\dot{A}^{i2}_{\rm cl}(\mathbf{x},t') - \nabla A^i_{\rm cl}(\mathbf{x},t')\cdot\nabla A^i_{\rm cl}(\mathbf{x},t')\big] \\ &= -\frac{1}{2}\int d\mathbf{x}\big[\dot{A}^i_{\rm cl}(\mathbf{x},t)A^i_{\rm cl}(\mathbf{x},t) - \dot{A}^i_{\rm cl}(\mathbf{x},0)A^i_{\rm cl}(\mathbf{x},0)\big] + \frac{1}{2}\int_0^t dt'\int d\mathbf{x}\, A^i_{\rm cl}\bigg[\frac{\partial^2}{\partial t'^2}A^i_{\rm cl}(\mathbf{x},t') - \nabla^2 A^i_{\rm cl}(\mathbf{x},t')\bigg] \\ &= -\frac{1}{2}\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)A^i_{\rm cl}(\mathbf{x},t) + \frac{1}{2}\int_0^t dt'\int d\mathbf{x}\, A^i_{\rm cl}(\mathbf{x},t')J^i(\mathbf{x},t').\end{aligned} \tag{A14}$$

In the second step we have used the dynamical equation in (4a). We are also assuming that the electron is initially at rest and starts moving after time 0, so we have the initial condition $\dot{A}^i_{\rm cl}(\mathbf{x},0) = \dot{A}^i_{\rm el}(\mathbf{x},0) = 0$. Therefore, the solution (A13) becomes

$$\begin{aligned}\langle\mathbf{A}|\psi_A,t\rangle = N\exp\bigg(&-\int d\mathbf{x}\,d\mathbf{x}'\big[A^i(\mathbf{x}) - A^i_{\rm cl}(\mathbf{x},t)\big]\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\big[A^i(\mathbf{x}') - A^i_{\rm cl}(\mathbf{x},t)\big] \\ &+ i\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)\big[A^i(\mathbf{x}) - A^i_{\rm cl}(\mathbf{x},t)\big]\bigg)\exp\bigg(\frac{i}{2}\int d\mathbf{x}\,\dot{A}^i_{\rm cl}(\mathbf{x},t)A^i_{\rm cl}(\mathbf{x},t) + \frac{i}{2}\int_0^t dt'\int d\mathbf{x}\,A^i_{\rm cl}(\mathbf{x},t')J^i(\mathbf{x},t')\bigg).\end{aligned} \tag{A15}$$

APPENDIX B: VECTOR POTENTIAL WAVE FUNCTION DESCRIBES A COHERENT STATE

We consider the Hamiltonian (3), written as

$$\begin{aligned}H &= \int d\mathbf{k}\,\omega a^{i\dagger}(\mathbf{k})a^i(\mathbf{k}) - \int d\mathbf{x}\,J^i(\mathbf{x},t)\hat{A}^i(\mathbf{x}) \\ &= \int d\mathbf{k}\,\omega a^{i\dagger}(\mathbf{k})a^i(\mathbf{k}) - \int \frac{d\mathbf{k}}{\sqrt{2\omega}}[\tilde{J}^{i*}(\mathbf{k},t)a^i(\mathbf{k}) + \tilde{J}^i(\mathbf{k},t)a^{i\dagger}(\mathbf{k})],\end{aligned} \tag{B1}$$

where

$$\hat{A}^i(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{1}{\sqrt{2\omega}}[a^i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^{i\dagger}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{B2a}$$

$$\tilde{J}^i(\mathbf{k},\mathbf{t}) \equiv \frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}\,J^i(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{B2b}$$

We will look for a solution of Schrödinger's equation in the form of a coherent state

$$|\psi_A,t\rangle = \exp\bigg(\int d\mathbf{k}\,\alpha^i(\mathbf{k},t)a^{i\dagger}(\mathbf{k})\bigg)|0\rangle e^{c(t)}. \tag{B3}$$

Inserting (B3) into Schrödinger's equation with the Hamiltonian (B1) and utilizing $a^i(\mathbf{k})|\psi,t\rangle = \alpha^i(\mathbf{k},t)|\psi,t\rangle$, we obtain the two equations

$$i\dot{\alpha}^i(\mathbf{k},t) = \omega\alpha^i(\mathbf{k},t) - \frac{1}{\sqrt{2\omega}}\tilde{J}^i(\mathbf{k},t), \tag{B4a}$$

$$i\dot{c}(t) = -\int d\mathbf{k}\frac{1}{\sqrt{2\omega}}\tilde{J}^{i*}(\mathbf{k},t)\alpha^i(\mathbf{k},t), \tag{B4b}$$

with solutions

$$\begin{aligned}\alpha^i(\mathbf{k},t) &= i\frac{1}{\sqrt{2\omega}}\int_0^t dt' e^{-i\omega(t-t')}\tilde{J}^i(\mathbf{k},t') \\ &= \frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\bigg[\sqrt{\frac{\omega}{2}}A^i_{\rm cl}(\mathbf{x},t) + i\frac{1}{\sqrt{2\omega}}\dot{A}^i_{\rm cl}(\mathbf{x},t)\bigg],\end{aligned} \tag{B5a}$$

$$\begin{aligned}c(t) &= -\int\frac{d\mathbf{k}}{2\omega}\int_0^t dt'\int_0^{t'} dt'' e^{-i\omega(t'-t'')}\tilde{J}^{i*}(\mathbf{k},t')\tilde{J}^i(\mathbf{k},t'') \\ &= \frac{i}{2}\int_0^t dt'\int d\mathbf{x}\,A^i_{\rm cl}(\mathbf{x},t')J^i(\mathbf{x},t') - \frac{1}{2}\int d\mathbf{k}|\alpha^i(\mathbf{k},t)|^2,\end{aligned} \tag{B5b}$$

where we have used (4b) with $c^i(\mathbf{k}) = 0$, i.e., for simplicity, we have assumed that all classical currents initially vanish, so $|0\rangle$ is the initial state. Thus, the solenoid current must be established first and enough time let lapse for the constant vector potential in the neighborhood of the electron to be set up, before the electron moves.

To prove that this solution is identical to the previously obtained wave function for the vector potential (A13), we note that (B3) is an overcomplete set of vectors if we regard $A^i_{\rm cl}(\mathbf{x},t)$ and $\dot{A}^i_{\rm cl}(\mathbf{x},t)$ in (B5a) abstractly, as arbitrarily choosable functions. Thus, our proof is complete if we show that the scalar product of any two of these vectors calculated using (B3) is the same as the scalar product of those vectors using (A15).

Denoting the two vectors by the subscripts R and L, we have

$$
\begin{aligned}
{}_L\langle\psi_A,t|\psi_A,t\rangle_R &= \exp\left(\int d\mathbf{k}\,\alpha^{i*}_L(\mathbf{k},t)\alpha^i_R(\mathbf{k},t)\right)e^{c^*_R(t)+c_L(t)}\\
&= \exp\Bigg[\int d\mathbf{k}\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}' e^{i\mathbf{k}\cdot\mathbf{x}'}\left(\sqrt{\frac{\omega}{2}}A^i_L(\mathbf{x}',t) - i\frac{1}{\sqrt{2\omega}}\dot{A}^i_L(\mathbf{x}',t)\right)\frac{1}{(2\pi)^{3/2}}\\
&\quad\times\int d\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\sqrt{\frac{\omega}{2}}A^i_R(\mathbf{x},t) + i\frac{1}{\sqrt{2\omega}}\dot{A}^i_R(\mathbf{x},t)\right)\Bigg]\\
&\quad\times\exp\Bigg[-\frac{1}{2}\int d\mathbf{k}\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}' e^{i\mathbf{k}\cdot\mathbf{x}'}\left(\sqrt{\frac{\omega}{2}}A^i_R(\mathbf{x}',t) - i\frac{1}{\sqrt{2\omega}}\dot{A}^i_R(\mathbf{x}',t)\right)\frac{1}{(2\pi)^{3/2}}\\
&\quad\times\int d\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\sqrt{\frac{\omega}{2}}A^i_R(\mathbf{x},t) + i\frac{1}{\sqrt{2\omega}}\dot{A}^i_R(\mathbf{x},t)\right)\Bigg]\\
&\quad\times\exp\Bigg[-\frac{1}{2}\int d\mathbf{k}\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}' e^{i\mathbf{k}\cdot\mathbf{x}'}\left(\sqrt{\frac{\omega}{2}}A^i_L(\mathbf{x}',t) - i\frac{1}{\sqrt{2\omega}}\dot{A}^i_L(\mathbf{x}',t)\right)\frac{1}{(2\pi)^{3/2}}\\
&\quad\times\int d\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}[\sqrt{\frac{\omega}{2}}A^i_L(\mathbf{x},t) + i\frac{1}{\sqrt{2\omega}}\dot{A}^i_L(\mathbf{x},t)]\Bigg]\\
&\quad\times\exp\left(\frac{i}{2}\int_0^t dt'\int d\mathbf{x}\,A^i_R(\mathbf{x},t')J^i_R(\mathbf{x},t') - \frac{i}{2}\int_0^t dt'\int d\mathbf{x}\,A^i_L(\mathbf{x},t')JL^i(\mathbf{x},t')\right)\\
&= \exp\left(-\frac{1}{4}\int d\mathbf{x}'\int d\mathbf{x}\big[A^i_L(\mathbf{x}',t) - A^i_R(\mathbf{x}',t)\big]\big[A^i_L(\mathbf{x},t) - A^i_R(\mathbf{x},t)\big]\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}']}\right)\\
&\quad\times\exp\left(-\frac{1}{4}\int d\mathbf{x}'\int d\mathbf{x}\big[\dot{A}^i_L(\mathbf{x}',t) - \dot{A}^i_R(\mathbf{x}',t)\big]\big[\dot{A}^i_L(\mathbf{x},t) - \dot{A}^i_R(\mathbf{x},t)\big]\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega^{-1}e^{i\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}']}\right)\\
&\quad\times\exp\left(\frac{i}{2}\int d\mathbf{x}[A^i_L(\mathbf{x},t)\dot{A}^i_R(\mathbf{x},t) - A^i_R(\mathbf{x},t)\dot{A}^i_L(\mathbf{x},t)]\right)\\
&\quad\times\exp\left(\frac{i}{2}\int_0^t dt'\int d\mathbf{x}\big[A^i_R(\mathbf{x},t')J^i_R(\mathbf{x},t') - A^i_L(\mathbf{x},t')J^i_L(\mathbf{x},t')\big]\right).
\end{aligned}\tag{B6}
$$

On the other hand, from (A15) we have

$$
\begin{aligned}
{}_L\langle\psi_A,t|\psi_A,t\rangle_R &= \int DA\,{}_L\langle\psi_A,t|\psi_A,t\rangle_R\\
&= \int DA\exp\left(\int d\mathbf{x}\,d\mathbf{x}'\big[A^i(\mathbf{x}) - A^i_R(\mathbf{x},t)\big]\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\big[A^i(\mathbf{x}') - A^i_{R,cl}(\mathbf{x}',t)\big]\right)\\
&\quad\times\exp\left(i\int d\mathbf{x}'\dot{A}^i_R(\mathbf{x},t)\big[A^i(\mathbf{x}) - A^i_R(\mathbf{x},t)\big]\right)\\
&\quad\times\exp\left(-\int d\mathbf{x}\,d\mathbf{x}'\big[A^i(\mathbf{x}) - A^i_L(\mathbf{x},t)\big]\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\big[A^i(\mathbf{x}') - A^i_L(\mathbf{x}',t)\big]\right)\\
&\quad\times\exp\left(-i\int d\mathbf{x}'\dot{A}^i_L(\mathbf{x},t)\big[A^i(\mathbf{x}) - A^i_L(\mathbf{x},t)\big]\right)\\
&\quad\times\exp\left(\frac{i}{2}\int d\mathbf{x}\big[\dot{A}^i_R(\mathbf{x},t)A^i_R(\mathbf{x},t) - \dot{A}^i_L(\mathbf{x},t)A^i_L(\mathbf{x},t\big] + \frac{i}{2}\int_0^t dt'\int d\mathbf{x}\big[A^i_R(\mathbf{x},t')J^i_R(\mathbf{x},t') - A^i_L(\mathbf{x},t')J^i_L(\mathbf{x},t')\big]\right).
\end{aligned}\tag{B7}
$$

We can apply

$$\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} dx\, e^{-(x-a)^2}e^{-(x-b)^2}e^{ip^1(x-a)}e^{-ip^2(x-b)} = e^{-[(a-b)^2/2]}e^{-[(p^1-p^2)^2/8]}e^{-i[(p^1+p^2)(a-b)/2]} \tag{B8}$$

to Eq. (B7) by diagonalizing its Gaussian exponents. The result is

$$\begin{aligned}{}_L\langle\psi_A,t|\psi_A,t\rangle_R &= \exp\left(-\frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\big[A^i_R(\mathbf{x},t)-A^i_L(\mathbf{x},t)\big]B(\mathbf{x}-\mathbf{x}')\big[A^i_R(\mathbf{x}',t)-A^i_L(\mathbf{x}',t)\big]\right)\\
&\times\exp\left(-\frac{1}{8}\int d\mathbf{x}\,d\mathbf{x}'\big[\dot A^i_R(\mathbf{x},t)-\dot A^i_L(\mathbf{x},t)\big]B^{-1}(\mathbf{x}-\mathbf{x}')\big[\dot A^i_R(\mathbf{x}',t)-\dot A^i_L(\mathbf{x}',t)\big]\right)\\
&\times\exp\left(-\frac{i}{2}\int d\mathbf{x}\big[\dot A^i_{R,\mathrm{cl}}(\mathbf{x},t)+\dot A^i_{L,\mathrm{cl}}(\mathbf{x},t)\big]\big[A^i_{R,\mathrm{cl}}(\mathbf{x},t)-A^i_{L,\mathrm{cl}}(\mathbf{x},t)\big]\right)\\
&\times\exp\left(\frac{i}{2}\int d\mathbf{x}\big[\dot A^i_R(\mathbf{x},t)A^i_R(\mathbf{x},t)-\dot A^i_L(\mathbf{x},t)A^i_L(\mathbf{x},t\big]+\frac{i}{2}\int_0^t dt'\int d\mathbf{x}\big[A^i_R(\mathbf{x},t')J^i_R(\mathbf{x},t')-A^i_L(\mathbf{x},t')J^i_L(\mathbf{x},t')\big]\right),\end{aligned} \tag{B9}$$

where $B(\mathbf{x}-\mathbf{x}')$ is given by (A6) and $B^{-1}(\mathbf{x}-\mathbf{x}') \equiv \frac{2}{(2\pi)^3}\int d\mathbf{k}\,\omega^{-1}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$. Combining terms on the third and fourth lines of (B9), we see that expressions (B6) and (B9) are identical.

APPENDIX C: OVERLAP INTEGRAL

We discuss here the contribution to the overlap integral $\langle\psi_{A,L},t|\psi_{A,R},t\rangle$ of the first exponential in Eq. (6). The amplitude and phase are given by the first three lines of Eq. (B9).

1. Phase

The third line of (B9) contains the phase

$$-\frac{1}{2}\int d\mathbf{x}\big[\dot A^i_{R,\mathrm{el}}(\mathbf{x},t)+\dot A^i_{L,\mathrm{el}}(\mathbf{x},t)\big]\big[A^i_{R,\mathrm{el}}(\mathbf{x},t)-A^i_{L,\mathrm{el}}(\mathbf{x},t)\big]. \tag{C1}$$

Because the left and right traverses have the same constant solenoid vector potential $A^i_{R,\mathrm{sol}}(\mathbf{x}) = A^i_{L,\mathrm{sol}}(\mathbf{x},t)$, this phase has been expressed in terms of the electron's classical vector potential alone. Under rotation of the physical situation about the y axis by 180°, $A^i_{L,\mathrm{el}}(\mathbf{x},t) \leftrightarrow A^i_{R,\mathrm{el}}(\mathbf{x},t)$ and so the integrand of (C1) changes sign. Since this is achieved by a coordinate transformation, Eq. (C1) is equal to its negative and therefore vanishes. Therefore, there is no phase shift contribution by the first exponential in (6).

2. Amplitude: Physical nature

Now consider the amplitude of the overlap integral $\equiv e^{-a(t)}$, given by the first two lines in (B9):

$$a(t) = \frac{1}{4}\int d\mathbf{x}\,d\mathbf{x}'A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}A^i_{RL}(\mathbf{x}',t)+\frac{1}{4}\int d\mathbf{x}\,d\mathbf{x}'\dot A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega^{-1}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\dot A^i_{RL}(\mathbf{x}',t), \tag{C2}$$

where $A^i_{RL}(\mathbf{x},t) \equiv A^i_R(\mathbf{x},t) - A^i_L(\mathbf{x},t)$. Suppose that $A^i_{RL}(\mathbf{x},t)$ is generated by a current for $0 \leqslant t \leqslant T$, but the current vanishes for $t > T$, so the field generated during $0 \leqslant t \leqslant T$ propagates freely for $t > T$. Then we can show that $a(t)$ is a constant of the motion for $t > T$ as follows:

$$\begin{aligned}\frac{d}{dt}a(t) &= \frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\dot A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}A^i_{RL}(\mathbf{x}',t)\\
&\quad+\frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\ddot A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega^{-1}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\dot A^i_{RL}(\mathbf{x}',t)\\
&= \frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\dot A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}A^i_{RL}(\mathbf{x}',t)\\
&\quad+\frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\nabla^2 A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega^{-1}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\dot A^i_{RL}(\mathbf{x}',t)\end{aligned}$$

$$= \frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\dot{A}^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}A^i_{RL}(\mathbf{x}',t)$$
$$-\frac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'A^i_{RL}(\mathbf{x},t)\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\dot{A}^i_{RL}(\mathbf{x}',t) = 0 \tag{C3}$$

where, in going from the second line to the fourth line, the free propagation of $A^i_{RL}(\mathbf{x},t)$ has been utilized.

What is this constant of the motion? Writing the free field in terms of its Fourier components as

$$A^i_{RL}(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{1}{\sqrt{2\omega}}[a^i(k)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + a^{i*}(k)e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}],$$

we find

$$a(t) = \int d\mathbf{k}a^{i*}(\mathbf{k})a^i(\mathbf{k}).$$

Were $a^i(\mathbf{k}),a^{i*}(\mathbf{k})$ annihilation and creation operators instead of c-number amplitudes, $a(t)$ would be the photon number operator. So we may think of $a(t)$ as a classical analog of the difference of the number of photons for the left and right traverses.

3. Amplitude: Divergence

Expressing the vector potential in terms of the current using the second line of (4b) and performing the integrals over $\mathbf{x}$ and $\mathbf{x}'$ and then over the δ functions, we obtain

$$a(t) = \frac{1}{4}\frac{1}{(2\pi)^3}\int_0^t dt_1\int_0^t dt_2\int d\mathbf{x}_1 d\mathbf{x}_2\int d\mathbf{k}\,e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\frac{1}{k}\cos k(t_1-t_2)$$
$$\times[\mathbf{J}_L(\mathbf{x}_1,t_1)-\mathbf{J}_R(\mathbf{x}_1,t_1)]\cdot[\mathbf{J}_L(\mathbf{x}_2,t_2)-\mathbf{J}_R(\mathbf{x}_2,t_2)]. \tag{C4}$$

Then, writing $\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2) = k|\mathbf{x}_1-\mathbf{x}_2|\cos\theta$, integrating over θ, writing $\sin k|\mathbf{x}_1-\mathbf{x}_2|\cos k(t_1-t_2)$ as the sum of sin functions, and using $\int_0^\infty dk\sin kz = \mathcal{P}\frac{1}{z}$, where $\mathcal{P}$ is the principal part, we obtain

$$a(t) = \frac{1}{4}\frac{1}{(2\pi)^2}\int_0^t dt_1\int_0^t dt_2\int d\mathbf{x}_1 d\mathbf{x}_2\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\mathcal{P}\left[\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|+(t_1-t_2)}+\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|-(t_1-t_2)}\right]$$
$$\times[\mathbf{J}_L(\mathbf{x}_1,t_1)-\mathbf{J}_R(\mathbf{x}_1,t_1)]\cdot[\mathbf{J}_L(\mathbf{x}_2,t_2)-\mathbf{J}_R(\mathbf{x}_2,t_2)]. \tag{C5}$$

Now, a classical electron orbiting counterclockwise in a half circle of radius R with speed u in the $z=0$ plane, starting at $\phi=-\pi/2$ at time 0, and ending at $\phi=\pi/2$ at time $T=\pi R/u$ provides the current $\mathbf{J}_R(\mathbf{x},t) = e\mathbf{u}_R(t)\delta(\mathbf{x}-\mathbf{R}_R(t))$, where $\mathbf{u}_R(t) = u[-\mathbf{i}\sin\phi_R(t)+\mathbf{j}\cos\phi_R(t)]$, $\mathbf{R}_R(t) = R[\mathbf{i}\cos\phi_R(t)+\mathbf{j}\sin\phi_R(t)]$, and $\phi_R(t) = -\frac{\pi}{2}+\frac{ut}{R}$.

Similarly, the electron orbiting clockwise starting at $\phi=3\pi/2$ [this is identified with $\phi=\pi/2$, so the discontinuity in angle is at $(x=0,y=-R)$] at time zero and ending also at $\phi=\pi/2$ at time $T=\pi R/u$ provides the current $\mathbf{J}_L(\mathbf{x},t) = e\mathbf{u}_L(t)\delta(\mathbf{x}-\mathbf{R}_L(t))$, where $\mathbf{u}_L(t) = u[-\mathbf{i}\sin\phi_L(t)+\mathbf{j}\cos\phi_L(t)]$, $\mathbf{R}_L(t) = R[\mathbf{i}\cos\phi_L(t)+\mathbf{j}\sin\phi_L(t)]$, and $\phi_L(t) = \frac{3\pi}{2}-\frac{ut}{R}$. Putting these currents into (C5) results in

$$a(t) = e^2u^2\frac{1}{(2\pi)^2}\int_0^t dt_1\int_0^t dt_2\mathcal{P}\left[\frac{\cos\frac{u}{R}(t_1-t_2)}{4R^2\sin^2\frac{u}{2R}(t_1-t_2)-(t_1-t_2)^2}-\frac{\cos\frac{u}{R}(t_1+t_2)}{4R^2\sin^2\frac{u}{2R}(t_1+t_2)-(t_1-t_2)^2}\right]. \tag{C6}$$

The integral of the first term in the bracket over (t_1-t_2) is divergent.

4. Removing the divergence: Smearing the charge

The divergence is due to the self-interacting terms in (C4) or (C5), $\mathbf{J}_L(\mathbf{x}_1,t_1)\mathbf{J}_L(\mathbf{x}_2,t_2)$ and $\mathbf{J}_R(\mathbf{x}_1,t_1)\mathbf{J}_R(\mathbf{x}_2,t_2)$. It occurs when $t_1=t_2$, when the point electron is superimposed upon itself. This suggests that the divergence might be ameliorated by smearing out the charge.

For example, if the current was due to a uniform "ball" of charge, we would set $\mathbf{J}(\mathbf{x},t) = e\mathbf{u}_R(t)\int d\mathbf{x}'\rho(\mathbf{x}')\delta(\mathbf{x}-\mathbf{R}_R(t)-\mathbf{x}')$ with $\rho(\mathbf{x}') = \frac{\Theta(\sigma-|\mathbf{x}'|)}{4\pi\sigma^3/3}$. We will adopt the simplest smearing, extending the point charge into a line charge in the z direction of length σ, setting $\rho(\mathbf{x}') = \delta(x')\delta(y')\Theta(\sigma-z')\Theta(z')/\sigma$. Then Eq. (C6) becomes, after the electron packets complete their traverse at time T, utilizing $uT=\pi R$ and putting in c and $\hbar$,

$$a(T) = \frac{e^2}{\hbar c}\frac{u^2}{4\sigma^2}\frac{1}{(2\pi)^2}\int_0^\sigma dz_1'\int_0^\sigma dz_2'\int_0^T dt_1\int_0^T dt_2$$
$$\times\mathcal{P}\left[\frac{\cos\frac{\pi}{T}(t_1-t_2)}{4(uT/\pi)^2\sin^2\frac{\pi}{2T}(t_1-t_2)+(z_1'-z_2')^2-c^2(t_1-t_2)^2}-\frac{\cos\frac{\pi}{T}(t_1+t_2)}{4(uT/\pi)^2\sin^2\frac{\pi}{2T}(t_1+t_2)+(z_1'-z_2')^2-c^2(t_1-t_2)^2}\right]$$

$$= \frac{e^2}{\hbar c}\frac{\beta^2}{4}\frac{1}{(2\pi)^2}\int_0^1 dz_1\int_0^1 dz_2\int_0^1 d\tau_1\int_0^1 d\tau_2$$
$$\times\,\mathcal{P}\left[\frac{\cos\pi(\tau_1-\tau_2)}{(\beta/\pi)^2\left[4\sin^2\frac{\pi}{2}(\tau_1-\tau_2)+\lambda^2(z_1-z_2)^2\right]-(\tau_1-\tau_2)^2}-\frac{\cos\pi(\tau_1+\tau_2)}{(\beta/\pi)^2\left[4\sin^2\frac{\pi}{2}(\tau_1+\tau_2)+\lambda^2(z_1-z_2)^2\right]-(\tau_1-\tau_2)^2}\right], \quad \text{(C7)}$$

where in the second expression we have changed to dimensionless variables $\tau_i \equiv t_i/T$, $z_i \equiv z_i'/\sigma$, $\beta \equiv u/c$, and $\lambda \equiv \sigma/R$.

Now we focus on the divergent first term in large square brackets of (C7), which we will call $a_1(T)$. Change variables to $\tau_\pm \equiv \tau_1 \pm \tau_2$ and $z_\pm \equiv z_1 \pm z_2$ and use

$$\int_0^1 d\tau_1\int_0^1 d\tau_2 = \frac{1}{2}\left[\int_{-1}^0 d\tau_-\int_{-\tau_-}^{2+\tau_-} d\tau_+ + \int_0^1 d\tau_-\int_{\tau_-}^{2-\tau_-} d\tau_+\right] = 2\int_0^1 d\tau_-[1-\tau_-],$$

and similarly for $z_\pm$, since the integral does not depend upon τ_+, z_+ and depends upon the square of each of τ_- and z_-. Then $a_1(T)$ is

$$a_1(T) = \frac{e^2}{\hbar c}\frac{\beta^2}{4}\frac{1}{(2\pi)^2}\int_0^1 d\tau_-[1-\tau_-]\cos\pi\tau_-\int_0^1 dz_-[1-z_-]\mathcal{P}\frac{1}{(\beta\lambda/\pi)^2 z_-^2-\left[\tau_-^2-(2\beta/\pi)^2\sin^2\frac{\pi}{2}\tau_-\right]}. \quad \text{(C8)}$$

We may neglect $(2\beta/\pi)^2\sin^2\frac{\pi}{2}\tau_-$ compared to τ_-^2, since it is a factor β^2 smaller.

For $|\alpha| < 1$, we have

$$\mathcal{P}\int_0^1 dz\frac{1}{z^2-\alpha^2} = \lim_{\epsilon\to 0}\left[-\int_0^{|\alpha|-\epsilon} dz\frac{1}{\alpha^2-z^2}+\int_{|\alpha|+\epsilon}^1 dz\frac{1}{z^2-\alpha^2}\right] = \frac{1}{2|\alpha|}\ln\frac{1-|\alpha|}{1+|\alpha|}, \quad \text{(C9a)}$$

$$\mathcal{P}\int_0^1 dz\frac{z}{z^2-\alpha^2} = \int_0^1 dz\frac{1}{z+|\alpha|}+|\alpha|\mathcal{P}\int_0^1 dz\frac{1}{z^2-\alpha^2} = \ln[1+|\alpha|]-\ln|\alpha|+\frac{1}{2}\ln\frac{1-|\alpha|}{1+|\alpha|}. \quad \text{(C9b)}$$

Since $|\alpha| = \frac{\pi}{\beta\lambda}|\tau_-|$, it is clear that the subsequent integral $\int_0^1 d\tau_- = \frac{\beta\lambda}{\pi}\int_0^{\pi/\beta\lambda} d|\alpha|$ no longer diverges.

The integral in (C8) is $\sim \frac{1}{\beta\lambda}$, so $a_1(T) \sim \frac{e^2}{\hbar c}\frac{u}{c}\frac{R}{\sigma}$. This result is used in Sec. III.

The second term in the large square brackets of (C7), which we call $a_2(T)$, is not divergent and does not need smearing, so we may set $\lambda = 0$. In addition, $a_2(T)$ is a factor β^2 smaller than $a_1(T)$.

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